

THE WEBSTER SCALAR CURVATURE FLOW ON CR SPHERE. PART II

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ABSTRACT. This is the second of two papers, in which we study the problem of prescribing Webster scalar curvature on the CR sphere as a given function f . Using the Webster scalar curvature flow, we prove an existence result under suitable assumptions on the Morse indices of f .

1. INTRODUCTION

Suppose (M, g_0) is a compact n -dimensional Riemannian manifold without boundary, where $n \geq 3$. Given a function f on M , the problem of prescribing scalar curvature is to find a metric g conformal to g_0 such that $R_g = f$. When f is constant, it is the Yamabe problem, which was solved by Trudinger [28], Aubin [1], and Schoen [26]. When (M, g_0) is the n -dimensional sphere S^n with g_0 being the standard metric in S^n , it is the so-called Nirenberg's problem and was studied in [4, 5, 6, 7, 21, 27]. Kazdan and Warner [21], using a clever integration by parts, found a necessary condition, which is now known as Kazdan-Warner condition. More precisely, they showed that if f can be prescribed as the scalar curvature of a metric $g = u^{\frac{4}{n-2}}g_0$, then

$$\int_{S^n} \langle \nabla_{g_0} f, \nabla_{g_0} x_i \rangle_{g_0} u^{\frac{2n}{n-2}} dV_{g_0} = 0 \quad \text{for } i = 1, 2, \dots, n+1,$$

where x_i is the coordinate function of \mathbb{R}^{n+1} restricted to S^n . Later, Chang and Yang [5] proved the following (see also [4]):

Theorem 1.1 (Chang-Yang). *Suppose that f is a smooth positive Morse function with only non-degenerate critical points and satisfies the degree condition:*

$$\sum_{\nabla_{g_0} f(x), \Delta_{g_0} f(x) < 0} (-1)^{\text{ind}(f, x)} \neq -1.$$

If $\|f - n(n+1)\|_{C^0(S^n)}$ is sufficiently small, then there exists a metric g conformal to g_0 such that its scalar curvature $R_g = f$.

Using the scalar curvature flow, Chen and Xu [2] was able to estimate how small $\|f - n(n+1)\|_{C^0(S^n)}$ should be. More precisely, they proved the following:

Theorem 1.2 (Chen and Xu [2]). *Suppose that f is a smooth positive function on the n -dimensional sphere S^n with only non-degenerate critical points with Morse indices $\text{ind}(f, x)$ and such that $\Delta_{g_0} f(x) \neq 0$ at any such point x . Let*

$$m_i = \#\{x \in S^n : \nabla_{g_0} f(x) = 0, \Delta_{g_0} f(x) < 0, \text{ind}(f, x) = n - i\}.$$

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Furthermore, suppose $\delta_n = 2^{\frac{2}{n}}$ if $3 \leq n \leq 4$ or $= 2^{\frac{2}{n-2}}$ for $n \geq 5$. If there is no solution with coefficient $k_i \geq 0$ to the system of equations

$$m_0 = 1 + k_0, m_i = k_{i-1} + k_i \text{ for } 1 \leq i \leq n, k_n = 0,$$

and f satisfies

$$\max_{S^n} f / \min_{S^n} f < \delta_n,$$

then f can be realized as the scalar curvature of some metric conformal to the standard metric g_0 .

In this paper, we are interested in the problem of prescribing Webster scalar curvature. More precisely, suppose (M, θ_0) is a compact strongly pseudoconvex CR manifold of real dimensional $2n+1$ with a given contact form θ_0 . We are interested in the following question: can we find a contact form θ conformal to θ_0 such that its Webster scalar curvature $R_\theta = f$? This has been studied in [9, 11, 14, 16, 23, 24, 25]. When f is constant, this is the CR Yamabe problem, which was solved by Jerison and Lee in [18, 19, 20], and by Gamara and Yacoub in [13, 15]. As an analogy of Nirenberg's problem, we want to study the problem of prescribing Webster scalar curvature on the CR sphere (S^{2n+1}, θ_0) .

From now on, we assume that f is a smooth positive Morse function on S^{2n+1} with only non-degenerate critical points in the sense that $\Delta_{\theta_0} f(x) \neq 0$ whenever $f'(x) = 0$. Here $f'(x)$ denotes the gradient of f with respect to the standard Riemannian metric on S^{2n+1} . In [23], Malchiodi and Uguzzoni proved the following:¹

Theorem 1.3 (Malchiodi and Uguzzoni [23]). *If f satisfies*

$$(1.1) \quad \sum_{f'(x)=0, \Delta_{\theta_0} f(x) < 0} (-1)^{\text{ind}(f,x)} \neq -1,$$

where $\text{ind}(f, x)$ denotes the Morse index of f at x , then f can be realized as the Webster scalar curvature of some contact form conformal to θ_0 , provided that f is sufficiently closed to the Webster scalar curvature of the standard contact form on S^{2n+1} in sup norm.

This is CR version of Theorem 1.1. It is important to know how large the difference in sup norm can possibly be. To answer this question, we follow the argument of Chen-Xu in [2] and consider the Webster scalar curvature flow. By using the Webster scalar curvature flow, we prove the following theorem, which is our main result:

Theorem 1.4. *Suppose that $n \geq 2$ and f is a smooth positive function on S^{2n+1} with only non-degenerate critical points with Morse indices $\text{ind}(f, x)$ and such that $\Delta_{\theta_0} f(x) \neq 0$ at any such point x . Let*

$$(1.2) \quad m_i = \#\{x \in S^{2n+1} : f'(x) = 0, \Delta_{\theta_0} f(x) < 0, \text{ind}(f, x) = 2n+1-i\}.$$

If there is no solution with coefficient $k_i \geq 0$ to the system of equations

$$(1.3) \quad m_0 = 1 + k_0, m_i = k_{i-1} + k_i \text{ for } 1 \leq i \leq 2n+1, k_{2n+1} = 0,$$

and f satisfies the simple bubble condition, namely

$$(sbc) \quad \max_{S^{2n+1}} f / \min_{S^{2n+1}} f < 2^{\frac{1}{n}},$$

¹Note that Theorem 1.3 in [23] was stated in terms of Heisenberg group \mathbb{H}^n . But one can easily see that the statement here is equivalent to theirs.

then f can be realized as the Webster scalar curvature of some contact form conformal to θ_0 .

We remark that Theorem 1.4 in fact implies Theorem 1.3. See the remark after the proof of Theorem 1.4 in section 5.

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2. THE WEBSTER SCALAR CURVATURE FLOW

Let θ_0 be the standard contact form on the sphere $S^{2n+1} = \{x = (x_1, \dots, x_{n+1}) : |x|^2 = 1\} \subset \mathbb{C}^{n+1}$, i.e.

$$\theta_0 = \sqrt{-1}(\bar{\partial} - \partial)|x|^2 = \sqrt{-1} \sum_{j=1}^{n+1} (x_j d\bar{x}_j - \bar{x}_j dx_j).$$

Then (S^{2n+1}, θ_0) is a compact strictly pseudoconvex CR manifold of real dimension $2n+1$. Suppose f is a smooth positive function on S^{2n+1} . Let $u_0 \in C^\infty(S^{2n+1})$ such that

$$(2.1) \quad \int_{S^{2n+1}} u_0^{2+\frac{2}{n}} dV_{\theta_0} = \int_{S^{2n+1}} dV_{\theta_0}.$$

We introduced the Webster scalar curvature flow in part I [17], which is defined as the evolution of the contact form $\theta = \theta(t)$, $t \geq 0$ as follows:

$$(2.2) \quad \frac{\partial}{\partial t} \theta = (\alpha f - R_\theta) \theta, \quad \theta|_{t=0} = u_0^{\frac{2}{n}} \theta_0,$$

where R_θ is the Webster scalar curvature of the contact form θ and $\alpha = \alpha(t)$ is given by

$$(2.3) \quad \alpha \int_{S^{2n+1}} f dV_\theta = \int_{S^{2n+1}} R_\theta dV_\theta.$$

If we write $\theta = u^{\frac{2}{n}} \theta_0$ where $u = u(t)$, then (2.2) is equivalent to the following evolution equation of the conformal factor u :

$$(2.4) \quad \frac{\partial u}{\partial t} = \frac{n}{2} (\alpha f - R_\theta) u, \quad u|_{t=0} = u_0.$$

Since $\theta = u^{\frac{2}{n}} \theta_0$, the Webster scalar curvature R_θ of θ satisfies the following CR Yamabe equation

$$(2.5) \quad R_\theta = u^{-(1+\frac{2}{n})} \left(-\left(2 + \frac{2}{n}\right) \Delta_{\theta_0} u + R_{\theta_0} u \right),$$

where $R_{\theta_0} = n(n+1)/2$ is the Webster scalar curvature of θ_0 .

We recall some of the results we have proved in part I. In [17], we established the long-time existence of the flow (2.2). See section 2.3 in [17]. Define

$$(2.6) \quad E(u) = \int_{S^{2n+1}} \left((2 + \frac{2}{n}) |\nabla_{\theta_0} u|_{\theta_0}^2 + R_{\theta_0} u^2 \right) dV_{\theta_0} = \int_{S^{2n+1}} R_{\theta} dV_{\theta}$$

where the last equality follows from (2.5). We also define

$$(2.7) \quad E_f(u) = \frac{E(u)}{(\int_{S^{2n+1}} f u^{2+\frac{2}{n}} dV_{\theta_0})^{\frac{n}{n+1}}}.$$

We have proved in part I that (see Proposition 2.2 in [17]):

Proposition 2.1. *The functional E_f is non-increasing along the flow (2.4). Indeed,*

$$\frac{d}{dt} E_f(u) = -n \int_{S^{2n+1}} (\alpha f - R_{\theta})^2 u^{2+\frac{2}{n}} dV_{\theta_0} / \left(\int_{S^{2n+1}} f u^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{n}{n+1}} \leq 0.$$

Recall the definition of the normalized contact form: For every smooth positive function $u(t)$, set $P(t) = \int_{S^{2n+1}} x u(t)^{2+\frac{2}{n}} dV_{\theta_0}$ where $x = (x_1, \dots, x_{n+1}) \in S^{2n+1} \subset \mathbb{C}^{n+1}$, and we define

$$(2.8) \quad \widehat{P(t)} = \frac{P(t)}{\|P(t)\|} \text{ if } \|P(t)\| \neq 0, \text{ otherwise } \widehat{P(t)} = P(t).$$

Clearly $\widehat{P(t)} \in S^{2n+1}$ smoothly depends on the time t if u does. There exists a family of conformal CR diffeomorphisms $\phi(t) : S^{2n+1} \rightarrow S^{2n+1}$ such that (see [12])

$$(2.9) \quad \int_{S^{2n+1}} x dV_h = (0, \dots, 0) \in \mathbb{C}^{n+1} \quad \text{for all } t > 0,$$

where the new contact form

$$(2.10) \quad h = h(t) = \phi(t)^*(\theta(t)) = v(t)^{2+\frac{2}{n}} \theta_0$$

is called the normalized contact form with $v = v(t) = (u(t) \circ \phi(t)) |\det(d\phi(t))|^{\frac{n}{2n+2}}$ and the volume form $dV_h = v(t)^{2+\frac{2}{n}} dV_{\theta_0}$. In fact, the conformal CR diffeomorphism may be represented as $\phi(t) = \phi_{p(t), r(t)} = \Psi \circ T_{p(t)} \circ D_{r(t)} \circ \pi$ for some $p(t) \in \mathbb{H}^n$ and $r(t) > 0$. Here the CR diffeomorphism $\pi : S^{2n+1} \setminus \{(0, \dots, 0, -1)\} \rightarrow \mathbb{H}^n$ is given by

$$(2.11) \quad \pi(x) = \left(\frac{x'}{1+x_{n+1}}, Re(\sqrt{-1} \frac{1-x_{n+1}}{1+x_{n+1}}) \right), x = (x', x_{n+1}) \in S^{2n+1},$$

where \mathbb{H}^n denotes the Heisenberg group, and $D_{\lambda}, T_{(z', \tau')} : \mathbb{H}^n \rightarrow \mathbb{H}^n$ are respectively the dilation and translation on \mathbb{H}^n given by

$$(2.12) \quad D_{\lambda}(z, \tau) = (\lambda z, \lambda^2 \tau) \text{ and } T_{(z', \tau')}(z, \tau) = (z + z', \tau + \tau' + 2Im(z' \cdot \bar{z})) \text{ for } (z, \tau) \in \mathbb{H}^n.$$

And $\Psi = \pi^{-1}$ is the inverse of π .

Meanwhile, the normalized function v satisfies

$$(2.13) \quad - (2 + \frac{2}{n}) \Delta_{\theta_0} v + R_{\theta_0} v = R_h v^{1+\frac{2}{n}},$$

where $R_h = R_{\theta} \circ \phi(t)$ is the Webster scalar curvature of the normalized contact form $h = h(t)$ in view of (2.10). Hereafter, we set $f_{\phi} = f \circ \phi$.

Let $\delta_n = \max_{S^{2n+1}} f / \min_{S^{2n+1}} f$. By assumption (sbc) in Theorem 1.4, we have $\delta_n < 2^{\frac{1}{n}}$. Then there exists $\epsilon_0 > 0$ such that

$$\frac{\delta_n^{\frac{n}{n+1}}}{2^{\frac{1}{n+1}}} = \frac{1 - \epsilon_0}{1 + \epsilon_0}.$$

In particular, $(1 + \epsilon_0)\delta_n^{\frac{n}{n+1}} < 2^{\frac{1}{n+1}}$. Set

$$(2.14) \quad \beta = (1 + \epsilon_0)Y(S^{2n+1}, \theta_0) \left(\min_{S^{2n+1}} f \right)^{-\frac{n}{n+1}}.$$

The following was proved in part I. See Theorem 4.7 in [17].

Theorem 2.2. *For any given u_0 satisfying (2.1) with $E_f(u_0) \leq \beta$ with β defined as in (2.14), consider the flow $\theta(t)$ defined in (2.2) with initial data u_0 . Let $\{t_k\}$ be a time sequence of the flow with $t_k \rightarrow \infty$ as $k \rightarrow \infty$. Let $\{\theta_k\}$ be the corresponding contact forms such that $\theta_k = u(t_k)^{\frac{2}{n}}\theta_0$. Assume that $\|R_{\theta_k} - R_\infty\|_{L^{p_1}(S^{2n+1}, \theta_k)} \rightarrow 0$ as $k \rightarrow \infty$ for some $p_1 > n + 1$ and a smooth function $R_\infty > 0$ defined on S^{2n+1} which satisfies the simple bubble condition (sbc):*

$$\frac{\max_{S^{2n+1}} R_\infty}{\min_{S^{2n+1}} R_\infty} < 2^{\frac{1}{n}}.$$

Then, up to a subsequence, either

- (i) $\{u_k\}$ is uniformly bounded in $S_2^p(S^{2n+1}, \theta_0)$ for some $p \in (n + 1, p_1)$. Furthermore, $u_k \rightarrow u_\infty$ in $S_2^p(S^{2n+1}, \theta_0)$ as $k \rightarrow \infty$, where $\theta_\infty = u_\infty^{\frac{2}{n}}\theta_0$ has Webster scalar curvature R_∞ , or
- (ii) let $h_k = \phi(t_k)^*(\theta_k) = v_k^{\frac{2}{n}}\theta_0$ be the associated sequence of the normalized contact forms satisfying $\int_{S^{2n+1}} x dV_{h_k} = (0, \dots, 0) \in \mathbb{C}^{n+1}$. Then, there exists $Q \in S^{2n+1}$ such that

$$(2.15) \quad dV_{\theta_k} \rightharpoonup \text{Vol}(S^{2n+1}, \theta_0)\delta_Q, \quad \text{as } k \rightarrow \infty$$

in the weak sense of measures. In addition, for any $\lambda \in (0, 1)$, we have

$$(2.16) \quad v_k \rightarrow 1 \text{ in } C_P^{1, \lambda}(S^{2n+1}) \quad \text{as } k \rightarrow \infty.$$

Here $C_P^{1, \lambda}(S^{2n+1})$ is the parabolic Hörmander Hölder spaces.

It follows from Theorem 2.2 that we have the following dichotomy: Either the flow converges in S_2^p for some $p > 2n + 2$, and in this case, f can be realized as the Webster scalar curvature of some contact form conformal to θ_0 thanks to Lemma 4.8 in [17], or the corresponding normalized flow $h(t)$ defined in (2.10) converges.

Starting from now, we will assume that, with the initial data $u_0 \in C_f^\infty$ where

$$u_0 \in C_f^\infty := \{u \in C_*^\infty : u > 0 \text{ and } E_f(u) \leq \beta\}$$

with β defined as (2.14) and

$$C_*^\infty := \left\{ 0 < u \in C^\infty(S^{2n+1}) : \theta = u^{\frac{2}{n}}\theta_0 \text{ satisfies } \int_{S^{2n+1}} u^{2+\frac{2}{n}} dV_{\theta_0} = \int_{S^{2n+1}} dV_{\theta_0} \right\},$$

the flow (2.4) does not converge and f cannot be realized as the Webster scalar curvature in the conformal class of θ_0 . So Theorem 2.2 can always be applied without further mention.

The following lemma was proved in part I. See Lemma 4.16 in [17].

Lemma 2.3. *Let $f : S^{2n+1} \rightarrow \mathbb{R}$ be a smooth positive non-degenerate Morse function satisfying the simple bubble condition (sbc):*

$$\frac{\max_{S^{2n+1}} f}{\min_{S^{2n+1}} f} < 2^{\frac{1}{n}}.$$

Suppose that f cannot be realized as the Webster scalar curvature of any contact form conformal to θ_0 . Let $u(t)$ be a smooth solution of (2.4) with initial data $u_0 \in C_f^\infty$. Then there exists a family of CR diffeomorphism $\phi(t)$ on S^{2n+1} with the normalized contact form $h(t) = v(t)^{\frac{2}{n}} \theta_0 = \phi(t)^(\theta(t))$ such that as $t \rightarrow \infty$*

$$v(t) \rightarrow 1, \quad h(t) \rightarrow \theta_0 \quad \text{in } C^{1,\gamma}(S^{2n+1})$$

for any $\gamma \in (0, 1)$, and $\phi(t) - \widehat{P(t)} \rightarrow 0$ in $L^2(S^{2n+1}, \theta_0)$. Moreover, as $t \rightarrow \infty$, we have

$$\|f \circ \phi(t) - f(\widehat{P(t)})\|_{L^2(S^{2n+1}, \theta_0)} \rightarrow 0 \quad \text{and} \quad \alpha(t)f(\widehat{P(t)}) \rightarrow R_{\theta_0}.$$

Here $\widehat{P(t)}$ is defined as in (2.8).

3. ANALYSIS ON THE VECTOR FIELD $\xi = (d\phi)^{-1} \frac{d\phi}{dt}$

3.1. Normalized curvature flow. Let us start with the normalized flow defined in (2.9) and (2.10), which satisfies (2.13). J. H. Cheng proved the following Kazdan-Warner type condition in [8]:

$$(3.1) \quad \int_{S^{2n+1}} \langle \nabla_{\theta_0} x, \nabla_{\theta_0} R_h \rangle_{\theta_0} dV_h = (0, \dots, 0) \quad \text{and} \quad \int_{S^{2n+1}} \langle \nabla_{\theta_0} \bar{x}, \nabla_{\theta_0} R_h \rangle_{\theta_0} dV_h = (0, \dots, 0),$$

where $x = (x_1, \dots, x_{n+1}) \in S^{2n+1} \subset \mathbb{C}^{n+1}$ and $\bar{x} = (\bar{x}_1, \dots, \bar{x}_{n+1})$.

For the corresponding conformal CR diffeomorphism of the normalized flow (2.9), we let $\phi(t) = (\phi_1(t), \dots, \phi_{n+1}(t)) \in S^{2n+1} \subset \mathbb{C}^{n+1}$. We define $\xi = (d\phi)^{-1} \frac{d\phi}{dt}$. Recall that $v(t) = (u(t) \circ \phi(t)) |\det(d\phi(t))|^{\frac{n}{2n+2}}$. Differentiating it with respect to t and using (2.4), we obtain

$$(3.2) \quad v_t = \frac{n}{2}(\alpha f_\phi - R_h)v + \frac{n}{2n+2}v \operatorname{div}'_h(\xi),$$

where div'_h is the subdivergence operator of type $(1, 0)$ with respect to the contact form h (see [8] for the definition). Differentiating (2.9) with respect to t and using (3.2), we get

$$(3.3) \quad \begin{aligned} (0, \dots, 0) &= \frac{d}{dt} \left(\int_{S^{2n+1}} x dV_h \right) = \frac{2n+2}{n} \left(\int_{S^{2n+1}} x v_t v^{1+\frac{2}{n}} dV_{\theta_0} \right) \\ &= (n+1) \int_{S^{2n+1}} x (\alpha f_\phi - R_h) dV_h + \int_{S^{2n+1}} x \operatorname{div}'_h(\xi) dV_h \\ &= (n+1) \int_{S^{2n+1}} x (\alpha f_\phi - R_h) dV_h + \int_{S^{2n+1}} x \operatorname{div}'_{\theta_0}(v^{2+\frac{2}{n}} \xi) dV_{\theta_0} \\ &= (n+1) \int_{S^{2n+1}} x (\alpha f_\phi - R_h) dV_h - \int_{S^{2n+1}} \xi dV_h. \end{aligned}$$

3.2. Cayley transform. The Cayley transform is the CR diffeomorphism $\pi : S^{2n+1} \setminus \{S\} \rightarrow \mathbb{H}^n$ given in (2.11), i.e.

$$\pi(x) = \left(\frac{x_1}{1+x_{n+1}}, \dots, \frac{x_n}{1+x_{n+1}}, \operatorname{Re}\left(\sqrt{-1}\frac{1-x_{n+1}}{1+x_{n+1}}\right) \right),$$

where $x = (x_1, \dots, x_n, x_{n+1}) \in S^{2n+1} \setminus \{S\}$, where $S = (0, \dots, 0, -1)$ and \mathbb{H}^n is the Heisenberg group. Note that $\Psi = \pi^{-1} : \mathbb{H}^n \rightarrow S^{2n+1}$ is given by

$$(3.4) \quad \Psi(z, \tau) = \left(\frac{2z}{1+|z|^2-\sqrt{-1}\tau}, \frac{1-|z|^2+\sqrt{-1}\tau}{1+|z|^2-\sqrt{-1}\tau} \right)$$

where $(z, \tau) \in \mathbb{H}^n \subset \mathbb{C}^n \times \mathbb{R}$. If we write $\Psi = (\Psi_1, \dots, \Psi_{n+1}) \in S^{2n+1} \subset \mathbb{C}^{n+1}$, and $(z, \tau) = (z_1, \dots, z_n, \tau) = (a_1 + \sqrt{-1}b_1, \dots, a_n + \sqrt{-1}b_n, \tau) \in \mathbb{H}^n \subset \mathbb{C}^n \times \mathbb{R}$, then

$$(3.5) \quad \begin{aligned} \frac{\partial \Psi_i}{\partial a_j} &= \frac{2\delta_{ij}}{1+|z|^2-\sqrt{-1}\tau} - \frac{4(a_i + \sqrt{-1}b_i)a_j}{(1+|z|^2-\sqrt{-1}\tau)^2}, \\ \frac{\partial \Psi_i}{\partial b_j} &= \frac{2\delta_{ij}\sqrt{-1}}{1+|z|^2-\sqrt{-1}\tau} - \frac{4(a_i + \sqrt{-1}b_i)b_j}{(1+|z|^2-\sqrt{-1}\tau)^2}, \quad \frac{\partial \Psi_i}{\partial \tau} = \frac{2\sqrt{-1}(a_i + \sqrt{-1}b_i)}{(1+|z|^2-\sqrt{-1}\tau)^2}, \end{aligned}$$

for $1 \leq i, j \leq n$, and

$$(3.6) \quad \begin{aligned} \frac{\partial \Psi_{n+1}}{\partial a_j} &= -\frac{4a_j}{(1+|z|^2-\sqrt{-1}\tau)^2}, \\ \frac{\partial \Psi_{n+1}}{\partial b_j} &= -\frac{4b_j}{(1+|z|^2-\sqrt{-1}\tau)^2}, \quad \frac{\partial \Psi_{n+1}}{\partial \tau} = \frac{2\sqrt{-1}}{(1+|z|^2-\sqrt{-1}\tau)^2}, \end{aligned}$$

for $1 \leq j \leq n$. Note that $X_j = \frac{\partial}{\partial a_j} + 2b_j \frac{\partial}{\partial \tau}$, $Y_j = \frac{\partial}{\partial b_j} - 2a_j \frac{\partial}{\partial \tau}$, $T = \frac{\partial}{\partial \tau}$, where $1 \leq j \leq n$, is a basis for the tangent space of \mathbb{H}^n . By (3.4), (3.5) and (3.6), we have

$$(3.7) \quad \begin{aligned} X_j(\Psi_i) &= \frac{2\delta_{ij}}{1+|z|^2-\sqrt{-1}\tau} - \frac{4z_i \bar{z}_j}{(1+|z|^2-\sqrt{-1}\tau)^2}, \\ Y_j(\Psi_i) &= \frac{2\sqrt{-1}\delta_{ij}}{1+|z|^2-\sqrt{-1}\tau} - \frac{4\sqrt{-1}z_i \bar{z}_j}{(1+|z|^2-\sqrt{-1}\tau)^2}, \\ T(\Psi_i) &= \frac{2\sqrt{-1}z_i}{(1+|z|^2-\sqrt{-1}\tau)^2} \end{aligned}$$

for $1 \leq i, j \leq n$, and

$$(3.8) \quad \begin{aligned} X_j(\Psi_{n+1}) &= -\frac{4\bar{z}_j}{(1+|z|^2-\sqrt{-1}\tau)^2}, \\ Y_j(\Psi_{n+1}) &= -\frac{4\sqrt{-1}\bar{z}_j}{(1+|z|^2-\sqrt{-1}\tau)^2}, \quad T(\Psi_{n+1}) = \frac{2\sqrt{-1}}{(1+|z|^2-\sqrt{-1}\tau)^2} \end{aligned}$$

for $1 \leq j \leq n$, where $\Psi = (\Psi_1, \dots, \Psi_{n+1})$. Recall that for $r > 0$ the dilation $D_r : \mathbb{H}^n \rightarrow \mathbb{H}^n$ and for $q = (z', \tau') \in \mathbb{H}^n$ the translation $T_{(z', \tau')} : \mathbb{H}^n \rightarrow \mathbb{H}^n$ are respectively given by

$$D_r(z, \tau) = (rz, r^2\tau) \text{ and } T_q(z, \tau) = (z + z', \tau + \tau' + 2\operatorname{Im}(z' \cdot \bar{z})) \text{ for } (z, \tau) \in \mathbb{H}^n.$$

If we define $\delta_{q,r} : \mathbb{H}^n \rightarrow \mathbb{H}^n$ as $\delta_{q,r} = T_q \circ D_r$, i.e.

$$(3.9) \quad \delta_{q,r}(z, \tau) = (rz + z', r^2\tau + \tau' + 2r\operatorname{Im}(z' \cdot \bar{z})) \text{ for } (z, \tau) \in \mathbb{H}^n,$$

then we have

$$\begin{aligned}
 d\delta_{q,r}(X_j) &= d\delta_{q,r} \left(\frac{\partial}{\partial a_j} + 2b_j \frac{\partial}{\partial \tau} \right) = r \left(\frac{\partial}{\partial a_j} + 2(rb_j + b'_j) \frac{\partial}{\partial \tau} \right), \\
 (3.10) \quad d\delta_{q,r}(Y_j) &= d\delta_{q,r} \left(\frac{\partial}{\partial b_j} - 2a_j \frac{\partial}{\partial \tau} \right) = r \left(\frac{\partial}{\partial b_j} - 2(ra_j + a'_j) \frac{\partial}{\partial \tau} \right), \\
 d\delta_{q,r}(T) &= d\delta_{q,r} \left(\frac{\partial}{\partial \tau} \right) = r^2 \frac{\partial}{\partial \tau},
 \end{aligned}$$

where $z' = (a'_1 + \sqrt{-1}b'_1, \dots, a'_n + \sqrt{-1}b'_n)$.

Now recall that the CR diffeomorphism $\phi = \phi(t) : S^{2n+1} \rightarrow S^{2n+1}$ is given by $\phi = \Psi \circ \delta_{q(t),r(t)} \circ \pi$ for some $q(t) = (z(t), \tau(t)) = (a_1(t) + \sqrt{-1}b_1(t), \dots, a_n(t) + \sqrt{-1}b_n(t), \tau(t)) \in \mathbb{H}^n$ and $r(t) > 0$. Therefore, by (3.9), we have

$$\begin{aligned}
 (3.11) \quad & \frac{d}{dt} \delta_{q(t),r(t)}(z, \tau) \\
 &= \frac{dr(t)}{dt} \sum_{j=1}^n \left[a_j \left(\frac{\partial}{\partial a_j} + 2(r(t)b_j + b_j(t)) \frac{\partial}{\partial \tau} \right) + b_j \left(\frac{\partial}{\partial b_j} - 2(r(t)a_j + a_j(t)) \frac{\partial}{\partial \tau} \right) \right] \\
 &+ 2 \frac{dr(t)}{dt} r(t) \tau \frac{\partial}{\partial \tau} + \sum_{j=1}^n \left[\frac{da_j(t)}{dt} \left(\frac{\partial}{\partial a_j} + 2(r(t)b_j + b_j(t)) \frac{\partial}{\partial \tau} \right) \right. \\
 &+ \frac{db_j(t)}{dt} \left(\frac{\partial}{\partial b_j} - 2(r(t)a_j + a_j(t)) \frac{\partial}{\partial \tau} \right) + 4r(t) \left(a_j \frac{db_j(t)}{dt} - b_j \frac{da_j(t)}{dt} \right) \frac{\partial}{\partial \tau} \\
 &\left. + 2 \left(a_j(t) \frac{db_j(t)}{dt} - b_j(t) \frac{da_j(t)}{dt} \right) \frac{\partial}{\partial \tau} \right] + \frac{d\tau(t)}{dt} \frac{\partial}{\partial \tau}.
 \end{aligned}$$

Using (3.10) and (3.11), we obtain

$$\begin{aligned}
 (3.12) \quad & (d\delta_{q(t),r(t)})^{-1} \left(\frac{d}{dt} \delta_{q(t),r(t)} \right) \\
 &= \frac{1}{r(t)} \frac{dr(t)}{dt} \sum_{j=1}^n (a_j X_j + b_j Y_j) + \frac{2\tau}{r(t)} \frac{dr(t)}{dt} T + \frac{1}{r(t)} \sum_{j=1}^n \left(\frac{da_j(t)}{dt} X_j + \frac{db_j(t)}{dt} Y_j \right) \\
 &+ \frac{4}{r(t)} \sum_{j=1}^n \left(a_j \frac{db_j(t)}{dt} - b_j \frac{da_j(t)}{dt} \right) T + \frac{2}{r(t)^2} \sum_{j=1}^n \left(a_j(t) \frac{db_j(t)}{dt} - b_j(t) \frac{da_j(t)}{dt} \right) T \\
 &+ \frac{1}{r(t)^2} \frac{d\tau(t)}{dt} T.
 \end{aligned}$$

Since $d\phi = d\Psi \circ d\delta_{q(t),r(t)} \circ d\pi$ and $\frac{d\phi}{dt} = d\Psi \circ \frac{d}{dt} (\delta_{q(t),r(t)} \circ \pi)$, we have

$$\begin{aligned}
 (3.13) \quad \xi &= (d\phi)^{-1} \frac{d\phi}{dt} = (d\pi)^{-1} \circ (d\delta_{q(t),r(t)})^{-1} \circ (d\Psi)^{-1} \left(d\Psi \circ \frac{d}{dt} \delta_{q(t),r(t)} \circ \pi \right) \\
 &= d\Psi \circ (d\delta_{q(t),r(t)})^{-1} \left(\frac{d}{dt} \delta_{q(t),r(t)} \circ \pi \right).
 \end{aligned}$$

Since $\xi = (\xi_1, \dots, \xi_{n+1})$, it follows from (3.7), (3.8), (3.12) and (3.13) that

$$\begin{aligned}
(3.14) \quad \xi_i &= \frac{1}{r(t)} \frac{dr(t)}{dt} \sum_{j=1}^n (a_j d\Psi_i(X_j) + b_j d\Psi_i(Y_j)) + \frac{2\tau}{r(t)} \frac{dr(t)}{dt} d\Psi_i(T) \\
&+ \frac{1}{r(t)} \sum_{j=1}^n \left(\frac{da_j(t)}{dt} d\Psi_i(X_j) + \frac{db_j(t)}{dt} d\Psi_i(Y_j) \right) \\
&+ \frac{4}{r(t)} \sum_{j=1}^n \left(a_j \frac{db_j(t)}{dt} - b_j \frac{da_j(t)}{dt} \right) d\Psi_i(T) \\
&+ \frac{1}{r(t)^2} \left[2 \sum_{j=1}^n \left(a_j(t) \frac{db_j(t)}{dt} - b_j(t) \frac{da_j(t)}{dt} \right) + \frac{d\tau(t)}{dt} \right] d\Psi_i(T) \\
&= \frac{1}{r(t)} \frac{dr(t)}{dt} \sum_{j=1}^n (a_j + \sqrt{-1}b_j) \left(\frac{2\delta_{ij}(1 + |z|^2 - \sqrt{-1}\tau) - 4z_i \bar{z}_j}{(1 + |z|^2 - \sqrt{-1}\tau)^2} \right) \\
&+ \frac{2\tau}{r(t)} \frac{dr(t)}{dt} \frac{2\sqrt{-1}z_i}{(1 + |z|^2 - \sqrt{-1}\tau)^2} \\
&+ \frac{1}{r(t)} \sum_{j=1}^n \left(\frac{da_j(t)}{dt} + \sqrt{-1} \frac{db_j(t)}{dt} \right) \left(\frac{2\delta_{ij}(1 + |z|^2 - \sqrt{-1}\tau) - 4z_i \bar{z}_j}{(1 + |z|^2 - \sqrt{-1}\tau)^2} \right) \\
&+ \frac{4}{r(t)} \sum_{j=1}^n \left(a_j \frac{db_j(t)}{dt} - b_j \frac{da_j(t)}{dt} \right) \frac{2\sqrt{-1}z_i}{(1 + |z|^2 - \sqrt{-1}\tau)^2} \\
&+ \frac{1}{r(t)^2} \left[2 \sum_{j=1}^n \left(a_j(t) \frac{db_j(t)}{dt} - b_j(t) \frac{da_j(t)}{dt} \right) + \frac{d\tau(t)}{dt} \right] \frac{2\sqrt{-1}z_i}{(1 + |z|^2 - \sqrt{-1}\tau)^2} \\
&= \frac{1}{r(t)} \frac{dr(t)}{dt} \frac{2z_i(1 - |z|^2)}{(1 + |z|^2 - \sqrt{-1}\tau)^2} + \frac{1}{r(t)} \frac{dr(t)}{dt} \frac{2\sqrt{-1}z_i\tau}{(1 + |z|^2 - \sqrt{-1}\tau)^2} \\
&+ \frac{1}{r(t)} \frac{dz_i(t)}{dt} \frac{2}{1 + |z|^2 - \sqrt{-1}\tau} - \frac{1}{r(t)} \sum_{j=1}^n \frac{dz_j(t)}{dt} \frac{4z_i \bar{z}_j}{(1 + |z|^2 - \sqrt{-1}\tau)^2} \\
&- \frac{1}{r(t)} \frac{8z_i \sqrt{-1}}{(1 + |z|^2 - \sqrt{-1}\tau)^2} \operatorname{Im} \left(\frac{dz(t)}{dt} \cdot z \right) \\
&+ \frac{1}{r(t)^2} \left[2 \operatorname{Im} \left(\frac{dz(t)}{dt} \cdot \overline{z(t)} \right) + \frac{d\tau(t)}{dt} \right] \frac{2\sqrt{-1}z_i}{(1 + |z|^2 - \sqrt{-1}\tau)^2} \\
&= \frac{1}{r(t)} \frac{dr(t)}{dt} \Psi_i \Psi_{n+1} + \frac{1}{r(t)} \frac{dz_i(t)}{dt} (1 + \Psi_{n+1}) - \frac{1}{r(t)} \sum_{j=1}^n \frac{dz_j(t)}{dt} \frac{(1 + \Psi_{n+1}) \Psi_i \bar{\Psi}_j}{1 + \Psi_{n+1}} \\
&- \frac{2}{r(t)} (1 + \Psi_{n+1}) \Psi_i \sqrt{-1} \operatorname{Im} \left(\frac{dz(t)}{dt} \cdot \left(\frac{\Psi_1}{1 + \Psi_{n+1}}, \dots, \frac{\Psi_n}{1 + \Psi_{n+1}} \right) \right) \\
&+ \frac{1}{r(t)^2} \left[2 \operatorname{Im} \left(\frac{dz(t)}{dt} \cdot \overline{z(t)} \right) + \frac{d\tau(t)}{dt} \right] \frac{\sqrt{-1}}{2} \Psi_i (1 + \Psi_{n+1})
\end{aligned}$$

for $1 \leq i \leq n$, and

(3.15)

$$\begin{aligned}
\xi_{n+1} &= \frac{1}{r(t)} \frac{dr(t)}{dt} \sum_{j=1}^n (a_j d\Psi_{n+1}(X_j) + b_j d\Psi_{n+1}(Y_j)) + \frac{2\tau}{r(t)} \frac{dr(t)}{dt} d\Psi_{n+1}(T) \\
&\quad + \frac{1}{r(t)} \sum_{j=1}^n \left(\frac{da_j(t)}{dt} d\Psi_{n+1}(X_j) + \frac{db_j(t)}{dt} d\Psi_{n+1}(Y_j) \right) \\
&\quad + \frac{4}{r(t)} \sum_{j=1}^n \left(a_j \frac{db_j(t)}{dt} - b_j \frac{da_j(t)}{dt} \right) d\Psi_{n+1}(T) \\
&\quad + \frac{1}{r(t)^2} \left[2 \sum_{j=1}^n \left(a_j(t) \frac{db_j(t)}{dt} - b_j(t) \frac{da_j(t)}{dt} \right) + \frac{d\tau(t)}{dt} \right] d\Psi_{n+1}(T) \\
&= -\frac{1}{r(t)} \frac{dr(t)}{dt} \sum_{j=1}^n \frac{4(a_j + \sqrt{-1}b_j)\bar{z}_j}{(1 + |z|^2 - \sqrt{-1}\tau)^2} + \frac{2\tau}{r(t)} \frac{dr(t)}{dt} \frac{2\sqrt{-1}}{(1 + |z|^2 - \sqrt{-1}\tau)^2} \\
&\quad - \frac{1}{r(t)} \sum_{j=1}^n \left(\frac{da_j(t)}{dt} + \sqrt{-1} \frac{db_j(t)}{dt} \right) \frac{4\bar{z}_j}{(1 + |z|^2 - \sqrt{-1}\tau)^2} \\
&\quad + \frac{4}{r(t)} \sum_{j=1}^n \left(a_j \frac{db_j(t)}{dt} - b_j \frac{da_j(t)}{dt} \right) \frac{2\sqrt{-1}}{(1 + |z|^2 - \sqrt{-1}\tau)^2} \\
&\quad + \frac{1}{r(t)^2} \left[2 \sum_{j=1}^n \left(a_j(t) \frac{db_j(t)}{dt} - b_j(t) \frac{da_j(t)}{dt} \right) + \frac{d\tau(t)}{dt} \right] \frac{2\sqrt{-1}}{(1 + |z|^2 - \sqrt{-1}\tau)^2} \\
&= -\frac{1}{r(t)} \frac{dr(t)}{dt} \frac{4|z|^2}{(1 + |z|^2 - \sqrt{-1}\tau)^2} + \frac{1}{r(t)} \frac{dr(t)}{dt} \frac{4\sqrt{-1}\tau}{(1 + |z|^2 - \sqrt{-1}\tau)^2} \\
&\quad - \frac{1}{r(t)} \sum_{j=1}^n \frac{dz_j(t)}{dt} \frac{4\bar{z}_j}{(1 + |z|^2 - \sqrt{-1}\tau)^2} - \frac{4}{r(t)} \frac{2\sqrt{-1}}{(1 + |z|^2 - \sqrt{-1}\tau)^2} \operatorname{Im} \left(\frac{d\bar{z}(t)}{dt} \cdot z \right) \\
&\quad + \frac{1}{r(t)^2} \left[2 \operatorname{Im} \left(\frac{dz(t)}{dt} \cdot \bar{z}(t) \right) + \frac{d\tau(t)}{dt} \right] \frac{2\sqrt{-1}}{(1 + |z|^2 - \sqrt{-1}\tau)^2} \\
&= \frac{1}{r(t)} \frac{dr(t)}{dt} (\Psi_{n+1}^2 - 1) - \frac{1}{r(t)} \sum_{j=1}^n \frac{dz_j(t)}{dt} \frac{\bar{\Psi}_j (1 + \Psi_{n+1})^2}{1 + \bar{\Psi}_{n+1}} \\
&\quad + \frac{2}{r(t)} (1 + \Psi_{n+1})^2 \sqrt{-1} \operatorname{Im} \left(\frac{d\bar{z}(t)}{dt} \cdot \left(\frac{\Psi_1}{1 + \Psi_{n+1}}, \dots, \frac{\Psi_n}{1 + \Psi_{n+1}} \right) \right) \\
&\quad + \frac{1}{r(t)^2} \left[2 \operatorname{Im} \left(\frac{dz(t)}{dt} \cdot \bar{z}(t) \right) + \frac{d\tau(t)}{dt} \right] \frac{\sqrt{-1}}{2} (1 + \Psi_{n+1})^2.
\end{aligned}$$

Thus, in our calculation, we may assume at time t , $q(t) = 0$ which simplify the calculation since otherwise it is the matter of the choice of the coordinates of S^{2n+1} . In doing so, we let

$$(3.16) \quad X = (X_1, \dots, X_{n+1}) = \int_{S^{2n+1}} \xi dV_{\theta_0},$$

and denote $r(t)^{-1}$ by ϵ . Then by symmetry, we obtain from (3.14) and (3.15) that

$$\begin{aligned}
(3.17) \quad X_i &= \epsilon \int_{S^{2n+1}} \left[\frac{dz_i(t)}{dt} - \frac{dz_i(t)}{dt} \frac{1 + \Psi_{n+1}}{1 + \bar{\Psi}_{n+1}} |\Psi_i|^2 \right. \\
&\quad \left. - 2(1 + \Psi_{n+1}) \Psi_i \sqrt{-1} \operatorname{Im} \left(\frac{\overline{dz_i(t)}}{dt} \frac{\Psi_i}{1 + \Psi_{n+1}} \right) \right] dV_{\theta_0} \\
&= \epsilon \int_{S^{2n+1}} \left[\frac{dz_i(t)}{dt} - \frac{dz_i(t)}{dt} \frac{1 + \Psi_{n+1}}{1 + \bar{\Psi}_{n+1}} |\Psi_i|^2 \right. \\
&\quad \left. - (1 + \Psi_{n+1}) \Psi_i \left(\frac{\overline{dz_i(t)}}{dt} \frac{\Psi_i}{1 + \Psi_{n+1}} - \frac{dz_i(t)}{dt} \frac{\bar{\Psi}_i}{1 + \bar{\Psi}_{n+1}} \right) \right] dV_{\theta_0} \\
&= \epsilon \frac{dz_i(t)}{dt} \int_{S^{2n+1}} dV_{\theta_0} - \epsilon \frac{d\bar{z}_i(t)}{dt} \int_{S^{2n+1}} \Psi_i^2 dV_{\theta_0} \\
&= \epsilon \frac{dz_i(t)}{dt} \int_{S^{2n+1}} dV_{\theta_0} - \epsilon \frac{d\bar{z}_i(t)}{dt} \int_{S^{2n+1}} \left(\operatorname{Re}(\Psi_i)^2 - \operatorname{Im}(\Psi_i)^2 + 2\sqrt{-1} \operatorname{Re}(\Psi_i) \operatorname{Im}(\Psi_i) \right) dV_{\theta_0} \\
&= \epsilon \operatorname{Vol}(S^{2n+1}, \theta_0) \frac{dz_i(t)}{dt}
\end{aligned}$$

for $1 \leq i \leq n$, and

$$\begin{aligned}
(3.18) \quad X_{n+1} &= \epsilon \int_{S^{2n+1}} \frac{dr(t)}{dt} \frac{\Psi_{n+1}^2 - 1}{2} dV_{\theta_0} \\
&\quad + \epsilon^2 \int_{S^{2n+1}} \left[2 \operatorname{Im} \left(\frac{dz(t)}{dt} \cdot \overline{z(t)} \right) + \frac{d\tau(t)}{dt} \right] \frac{\sqrt{-1}}{2} (1 + \Psi_{n+1})^2 dV_{\theta_0} \\
&= \frac{\epsilon}{2} \frac{dr(t)}{dt} \int_{S^{2n+1}} (\operatorname{Re}(\Psi_{n+1})^2 - \operatorname{Im}(\Psi_{n+1})^2 - 1) dV_{\theta_0} \\
&\quad + \frac{\epsilon^2 \sqrt{-1}}{2} \left[2 \operatorname{Im} \left(\frac{dz(t)}{dt} \cdot \overline{z(t)} \right) + \frac{d\tau(t)}{dt} \right] \int_{S^{2n+1}} (\operatorname{Re}(\Psi_{n+1})^2 - \operatorname{Im}(\Psi_{n+1})^2 + 1) dV_{\theta_0} \\
&= \frac{\epsilon}{2} \operatorname{Vol}(S^{2n+1}, \theta_0) \left(-\frac{dr(t)}{dt} + 2\sqrt{-1} \epsilon \operatorname{Im} \left(\frac{dz(t)}{dt} \cdot \overline{z(t)} \right) + \sqrt{-1} \epsilon \frac{d\tau(t)}{dt} \right).
\end{aligned}$$

Now we are going to get the estimate on the conformal vector field ξ .

Lemma 3.1. *There exists a constant $C > 0$ such that*

$$\|\xi\|_{L^\infty}^2 \leq C \int_{S^{2n+1}} (\alpha(t) f_\phi - R_h)^2 dV_h.$$

Proof. Note that $\|\Psi_i\|_{L^\infty} \leq 3$ for $i = 1, \dots, n+1$. Thus by (3.14) and (3.15), we have

$$(3.19) \quad \|\xi\|_{L^\infty} \leq C \left(\left| \epsilon \frac{dr(t)}{dt} \right| + \sum_{i=1}^n \left| \epsilon \frac{dz_i(t)}{dt} \right| + \left| 2\epsilon^2 \operatorname{Im} \left(\frac{dz(t)}{dt} \cdot \overline{z(t)} \right) + \epsilon^2 \frac{d\tau(t)}{dt} \right| \right)$$

for some constant C independent of t . By (3.17) and (3.18), we have

$$(3.20) \quad \begin{aligned} \epsilon \frac{dz_i(t)}{dt} &= \frac{1}{\text{Vol}(S^{2n+1}, \theta_0)} X_i, \quad \epsilon \frac{dr(t)}{dt} = -\frac{1}{\text{Vol}(S^{2n+1}, \theta_0)} (X_{n+1} + \bar{X}_{n+1}), \\ \sqrt{-1}\epsilon^2 \left[2\text{Im} \left(\frac{dz(t)}{dt} \cdot \overline{z(t)} \right) + \frac{d\tau(t)}{dt} \right] &= -\frac{1}{\text{Vol}(S^{2n+1}, \theta_0)} (X_{n+1} - \bar{X}_{n+1}). \end{aligned}$$

Hence, it follows from (3.19), (3.20) and Cauchy-Schwartz inequality that

$$(3.21) \quad \|\xi\|_{L^\infty} \leq C_0 \|X\|$$

for some constant C_0 independent of t . Here $\|X\|$ is the norm of the vector $X \in \mathbb{C}^{n+1}$, i.e. $\|X\|^2 = \sum_{i=1}^{n+1} |X_i|^2$. Combining (3.3) and (3.21), we get the following estimate:

$$(3.22) \quad \begin{aligned} \|\xi\|_{L^\infty} &\leq C_0 \|X\| \\ &\leq C_0 \left((n+1) \left| \int_{S^{2n+1}} x(\alpha(t)f_\phi - R_h) dV_h \right| + \left| \int_{S^{2n+1}} \xi(1 - v^{2+\frac{2}{n}}) dV_{\theta_0} \right| \right) \\ &\leq C_0 \left((n+1) \left| \int_{S^{2n+1}} x(\alpha(t)f_\phi - R_h) dV_h \right| + \text{Vol}(S^{2n+1}, \theta_0) \|\xi\|_{L^\infty} \|v^{2+\frac{2}{n}} - 1\|_{C^0} \right). \end{aligned}$$

Then by Lemma 2.3, $\|v^{2+\frac{2}{n}} - 1\|_{C^0} \rightarrow 0$ as $t \rightarrow \infty$. Hence there exists a $T > 0$ such that $C_0 \text{Vol}(S^{2n+1}, \theta_0) \|v^{2+\frac{2}{n}} - 1\|_{C^0} \leq 1/2$ if $t \geq T$. Hence, by (3.22), for all $t \geq T$ we have

$$\begin{aligned} \|\xi\|_{L^\infty} &\leq 2C_0(n+1) \left| \int_{S^{2n+1}} x(\alpha(t)f_\phi - R_h) dV_h \right| \\ &\leq 2C_0(n+1) \text{Vol}(S^{2n+1}, \theta_0)^{\frac{1}{2}} \left(\int_{S^{2n+1}} (\alpha(t)f_\phi - R_h)^2 dV_h \right)^{\frac{1}{2}} \end{aligned}$$

by Hölder's inequality. On the other hand, ξ is continuous on $S^{2n+1} \times [0, T]$. Setting $C_1 = \max_{(x,t) \in S^{2n+1} \times [0, T]} \|\xi\|_{L^\infty}^2$, we conclude that

$$\|\xi\|_{L^\infty}^2 \leq \frac{C_1}{\min_{t \in [0, T]} F_2(t)} F_2(t)$$

for all $t \leq T$. Here we observe that $F_2(t)$ can never be zero for any finite t , otherwise f could be realized as the Webster scalar curvature of some conformal contact form. Hence this lemma follows from these two estimates. \square

4. SPECTRAL DECOMPOSITION

For convenience, we denote

$$F_p(t) = \int_{S^{2n+1}} |R_\theta - \alpha f|^p dV_\theta \quad \text{and} \quad G_p(t) = \int_{S^{2n+1}} |\nabla_\theta(R_\theta - \alpha f)|_\theta^p dV_\theta$$

for $p \geq 1$. The following lemma was proved in part I. See Lemma 3.2 and 3.3 in [17].

Lemma 4.1. *For any $p < \infty$, there holds $F_p(t) \rightarrow 0$ as $t \rightarrow \infty$. There also holds $G_2(t) \rightarrow 0$ as $t \rightarrow \infty$.*

The following lemma was also proved in part I. See Lemma 5.1 in [17].

Lemma 4.2. *With error $o(1) \rightarrow 0$ as $t \rightarrow \infty$, there holds*

$$\frac{d}{dt}F_2(t) \leq (n+1+o(1))(nF_2(t) - 2G_2(t)) + o(1)F_2(t).$$

4.1. The shadow flow. From now on, we assume that $n \geq 2$, as in the assumption of Theorem 1.4. Recall Theorem 2.2, the center of mass $\Theta(t)$ of the contact form $\theta(t)$ is given approximately by

$$\Theta(t) = \int_{S^{2n+1}} \phi(t) dV_{\theta_0}$$

with $\widehat{\Theta(t)} = \frac{\Theta(t)}{\|\Theta(t)\|}$. For any given $t \geq 0$, rotate $\widehat{\Theta(t)}$ as the south pole, then the conformal CR diffeomorphism may be represented as $\phi(t) = \Psi \circ \delta_{q(t), r(t)} \circ \pi$ for some $q(t) \in \mathbb{H}^n$ and $r(t) > 0$. In the following lemma, we extend $f(\mu y) = f(y)$ for $0 < \mu < 1$, $y \in \mathbb{S}^{2n+1}$.

Lemma 4.3. *With a uniform constant $C > 0$, if one set $\epsilon = 1/r(t)$, then there holds*

$$\|f_\phi - f(\widehat{\Theta(t)})\|_{L^2(S^{2n+1}, \theta_0)} + \|\nabla_{\theta_0} f_\phi\|_{L^2(S^{2n+1}, \theta_0)} \leq C\epsilon.$$

Proof. We choose the coordinate at the point $\widehat{\Theta(t)}$ which can be represented as the north pole so that S^{2n+1} can be represented by Ψ , where $\Psi(z, \tau) = \pi^{-1}(z, \tau)$, $(z, \tau) \in \mathbb{H}^n$ defined in (3.4). For simplicity, we set $\epsilon(t) = \frac{1}{r(t)}$. Hence, by a calculation similar to (3.5) and (3.6) we have

$$\begin{aligned} \int_{S^{2n+1}} |\nabla_{\theta_0} \phi|_{\theta_0}^2 dV_{\theta_0} &= \int_{\mathbb{H}^n} |\nabla_{\Psi^*(\theta_0)}(\phi \circ \Psi)|_{\Psi^*(\theta_0)}^2 \left(\frac{4}{\tau^2 + (1 + |z|^2)^2} \right)^{n+1} dz d\tau \\ &= \int_{\mathbb{H}^n} \left(\frac{4n\epsilon^2}{(1 + \epsilon^2|z|^2)^2 + \epsilon^4\tau^2} \right) \left(\frac{4}{\tau^2 + (1 + |z|^2)^2} \right)^n dz d\tau \\ &\leq C\epsilon^2 \int_{B_{\epsilon^{-1}}(0)} \frac{dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^n} \\ &\quad + C\epsilon^{-2} \int_{\mathbb{H}^n \setminus B_{\epsilon^{-1}}(0)} \frac{dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} \\ &\leq C\epsilon^2 + C\epsilon^{2n-2} \leq C\epsilon, \end{aligned}$$

where we have used the estimates: For $0 \leq m \leq \frac{n}{2}$, we have

$$\begin{aligned}
(4.1) \quad & \int_{B_{\epsilon^{-1}}(0)} \frac{(\tau^2 + |z|^4)^m dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} \\
& \leq \int_{B_{\epsilon^{-1}}(0)} \frac{dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1-m}} = \int_{\{4\sqrt{\tau^2 + |z|^4} \leq \epsilon^{-1}\}} \frac{dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1-m}} \\
& \leq \int_{\{|z| \leq \epsilon^{-1}\}} \left(\int_{-\sqrt{\epsilon^{-4} - |z|^4}}^{\sqrt{\epsilon^{-4} - |z|^4}} \frac{d\tau}{1 + \tau^2} \right) \frac{dz}{(1 + |z|^2)^{2n-2m}} \\
& = \int_{\{|z| \leq \epsilon^{-1}\}} \left[\tan^{-1}(\tau) \right]_{-\sqrt{\epsilon^{-4} - |z|^4}}^{\sqrt{\epsilon^{-4} - |z|^4}} \frac{dz}{(1 + |z|^2)^{2n-2m}} \\
& \leq \pi \int_{\{|z| \leq \epsilon^{-1}\}} \frac{dz}{(1 + |z|^2)^{2n-2m}} = C \int_0^{\epsilon^{-1}} \frac{r^{2n-1} dr}{(1 + r^2)^{2n-2m}} \\
& = C \left(\int_0^1 \frac{r^{2n-1} dr}{(1 + r^2)^{2n-2m}} + \int_1^{\epsilon^{-1}} \frac{r^{2n-1} dr}{(1 + r^2)^{2n-2m}} \right) \\
& \leq C \left(\int_0^1 \frac{dr}{(1 + r^2)^{1-2m}} + \int_1^{\epsilon^{-1}} \frac{dr}{r^{2n-4m+1}} \right) = \begin{cases} C + C\epsilon^{2n-4m}, & \text{if } m < \frac{n}{2}; \\ C + C \log \epsilon, & \text{if } m = \frac{n}{2}, \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
(4.2) \quad & \int_{\mathbb{H}^n \setminus B_{\epsilon^{-1}}(0)} \frac{dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^k} = \int_{\{4\sqrt{\tau^2 + |z|^4} \geq \epsilon^{-1}\}} \frac{dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^k} \\
& \leq 2 \int_{\{|z| \geq \epsilon^{-1}\}} \left(\int_{\sqrt{\epsilon^{-4} - |z|^4}}^{\infty} \frac{d\tau}{1 + \tau^2} \right) \frac{dz}{(1 + |z|^2)^{2k-2}} \\
& \leq \pi \int_{\{|z| \geq \epsilon^{-1}\}} \frac{dz}{(1 + |z|^2)^{2k-2}} = C \int_{\epsilon^{-1}}^{\infty} \frac{r^{2n-1} dr}{(1 + r^2)^{2k-2}} \\
& \leq C \int_{\epsilon^{-1}}^{\infty} \frac{dr}{r^{4k-2n-3}} = O(\epsilon^{4k-2n-4}) \quad \text{if } 4k \geq 2n + 5.
\end{aligned}$$

Recall that $\Theta(t)$ is the average of $\phi(t)$, from the Poincaré-type inequality (see Theorem 3.20 in [10]), we have

$$\|\phi(t) - \Theta(t)\|_{L^2(S^{2n+1}, \theta_0)} \leq C \|\nabla_{\theta_0} \phi\|_{L^2(S^{2n+1}, \theta_0)} \leq C\epsilon.$$

Here we need the assumption $n \geq 2$ to conclude that $p = 2 < n + 1$ so that Theorem 3.20 in [10] can be applied. Hence, by the inequalities

$$|f_\phi - f(\widehat{\Theta(t)})| = |f_\phi - f(\Theta(t))| \leq \|\nabla f\|_{L^\infty} |\phi(t) - \Theta(t)|$$

and

$$|\nabla_{\theta_0} f_\phi| \leq \|\nabla f\|_{L^\infty} |\nabla_{\theta_0} \phi|,$$

the assertion follows. \square

Let $\{\varphi_i\}$ be an $L^2(S^{2n+1}, \theta_0)$ -orthonormal basis of eigenfunctions of $-\Delta_{\theta_0}$, satisfying $-\Delta_{\theta_0} \varphi_i = \lambda_i \varphi_i$ with eigenvalues $0 = \lambda_0 < \lambda_1 = \dots = \lambda_{2n+2} = \frac{n}{2} < \lambda_{2n+3} \leq$

... In fact, we can take with loss of generality

$$(4.3) \quad \varphi_i = \frac{1}{\sqrt{n+1}}x_i \quad \text{and} \quad \varphi_{n+1+i} = \frac{1}{\sqrt{n+1}}\bar{x}_i \quad \text{for } i = 1, \dots, n+1,$$

where $x = (x_1, \dots, x_{n+1})$ is the coordinates of \mathbb{C}^{n+1} restricted to S^{2n+1} . Now in terms of the orthonormal basis $\{\varphi_i^\theta\}, \{\varphi_i^h\}$ of the eigenfunctions of $-\Delta_\theta, -\Delta_h$ with the corresponding eigenvalues $\lambda_i^\theta, \lambda_i^h$ respectively, we expand

$$\alpha f - R_\theta = \sum_{i=0}^{\infty} \beta_\theta^i \varphi_i^\theta \quad \text{and} \quad \alpha f_\phi - R_h = \sum_{i=0}^{\infty} \beta_h^i \varphi_i^h,$$

with coefficients

$$(4.4) \quad \beta_h^i = \int_{S^{2n+1}} (\alpha f_\phi - R_h) \varphi_i^h dV_h = \int_{S^{2n+1}} (\alpha f - R_\theta) \varphi_i^\theta dV_\theta = \beta_\theta^i$$

for all $i \in \mathbb{N}$. First notice that we always have $\beta_0^\theta = 0$ in view of (2.3). It is well known that $\varphi_i^h = \varphi_i^\theta \circ \phi$, which implies (4.4) and $\lambda_i^\theta = \lambda_i^h$ for all $i \in \mathbb{N}$.

Lemma 4.4. *As $t \rightarrow \infty$, we have $\lambda_i^\theta = \lambda_i^h \rightarrow \lambda_i$ and we can choose φ_i such that $\varphi_i^h \rightarrow \varphi_i$ in $L^2(S^{2n+1}, \theta_0)$ for all $i \in \mathbb{N}$.*

Since the proof is essentially the same as the proof of Lemma 5.2 in part I, we omit the proof and refer the reader to [17]. Now we define

$$(4.5) \quad b = (b^1, \dots, b^{2n+2}) = \int_{S^{2n+1}} (x, \bar{x}) (\bar{R}_h - R_h) dV_h$$

where $x = (x_1, \dots, x_{n+1}) \in S^{2n+1} \subset \mathbb{C}^{n+1}$ and $\bar{x} = (\bar{x}_1, \dots, \bar{x}_{n+1})$. That is,

$$b^i = \int_{S^{2n+1}} x_i (\alpha f_\phi - R_h) dV_h \quad \text{and} \quad b^{n+1+i} = \int_{S^{2n+1}} \bar{x}_i (\alpha f_\phi - R_h) dV_h \quad \text{for } 1 \leq i \leq n+1.$$

For brevity, set $B = \sqrt{n+1}b$, $\beta_\theta = (\beta_\theta^1, \dots, \beta_\theta^{2n+2})$, then by (4.3), (4.4) and Lemma 4.4

$$(4.6) \quad \begin{aligned} |B^i - \beta_\theta^i| &= |\sqrt{n+1}b^i - \beta_\theta^i| \\ &= \left| \sqrt{n+1} \int_{S^{2n+1}} x_i (\alpha f_\phi - R_h) dV_h - \int_{S^{2n+1}} \varphi_i^h (\alpha f_\phi - R_h) dV_h \right| \\ &= \left| \int_{S^{2n+1}} (\varphi_i - \varphi_i^h) (\alpha f_\phi - R_h) dV_h \right| \\ &\leq \|\varphi_i - \varphi_i^h\|_{L^2(S^{2n+1}, h)} \|\alpha f_\phi - R_h\|_{L^2(S^{2n+1}, h)} \\ &\leq C \|\varphi_i - \varphi_i^h\|_{L^2(S^{2n+1}, \theta_0)} F_2(h(t))^{\frac{1}{2}} = o(1) F_2(t)^{\frac{1}{2}} \end{aligned}$$

for $i = 1, 2, \dots, 2n+2$, where $o(1) \rightarrow 0$ as $t \rightarrow \infty$.

Lemma 4.5. *With error $o(1) \rightarrow 0$ as $t \rightarrow \infty$, there holds*

$$\frac{dB(t)}{dt} = o(1) F_2(t)^{\frac{1}{2}}.$$

Proof. By (2.13), we have

$$b = \int_{S^{2n+1}} (x, \bar{x}) \alpha f_\phi dV_h - \int_{S^{2n+1}} (x, \bar{x}) v \left(-(2 + \frac{2}{n}) \Delta_{\theta_0} v + R_{\theta_0} v \right) dV_{\theta_0}.$$

Thus

$$\begin{aligned}
(4.7) \quad \frac{db}{dt} &= \alpha_t \int_{S^{2n+1}} (x, \bar{x}) f_\phi dV_h + \int_{S^{2n+1}} (x, \bar{x}) \alpha df_\phi \cdot \xi dV_h \\
&\quad + \left(2 + \frac{2}{n}\right) \int_{S^{2n+1}} (x, \bar{x}) \alpha f_\phi v^{1+\frac{2}{n}} v_t dV_{\theta_0} \\
&\quad - \int_{S^{2n+1}} (x, \bar{x}) v_t \left(-\left(2 + \frac{2}{n}\right) \Delta_{\theta_0} v + R_{\theta_0} v \right) dV_{\theta_0} \\
&\quad - \int_{S^{2n+1}} (x, \bar{x}) v \left(-\left(2 + \frac{2}{n}\right) \Delta_{\theta_0} v_t + R_{\theta_0} v_t \right) dV_{\theta_0} \\
&= \alpha_t \int_{S^{2n+1}} (x, \bar{x}) f_\phi dV_h + \int_{S^{2n+1}} (x, \bar{x}) \alpha df_\phi \cdot \xi dV_h \\
&\quad + \left(2 + \frac{2}{n}\right) \int_{S^{2n+1}} (x, \bar{x}) \alpha f_\phi v^{1+\frac{2}{n}} v_t dV_{\theta_0} - 2R_{\theta_0} \int_{S^{2n+1}} (x, \bar{x}) v v_t dV_{\theta_0} \\
&\quad + \left(2 + \frac{2}{n}\right) \int_{S^{2n+1}} (x, \bar{x}) (v_t \Delta_{\theta_0} v + v \Delta_{\theta_0} v_t) dV_{\theta_0}.
\end{aligned}$$

We are going to estimate the terms on the right hand side of (4.7). By (2.9) and Lemma 2.3, and by (3.4) in [17], the first term on the right hand side of (4.7) can be bounded by

$$\alpha_t \int_{S^{2n+1}} (x, \bar{x}) f_\phi dV_h = \alpha_t \int_{S^{2n+1}} (x, \bar{x}) (f_\phi - f(\widehat{P(t)})) dV_h = o(1) F_2(t)^{\frac{1}{2}}.$$

Observe that by (4.3) and integration by parts, the last four terms on the right hand side of (4.7) can be rewritten as

$$\begin{aligned}
(4.8) \quad &\int_{S^{2n+1}} (x, \bar{x}) \alpha df_\phi \cdot \xi dV_h + \left(2 + \frac{2}{n}\right) \int_{S^{2n+1}} (x, \bar{x}) \alpha f_\phi v^{1+\frac{2}{n}} v_t dV_{\theta_0} \\
&- 2R_{\theta_0} \int_{S^{2n+1}} (x, \bar{x}) v v_t dV_{\theta_0} + \left(2 + \frac{2}{n}\right) \int_{S^{2n+1}} (x, \bar{x}) (v_t \Delta_{\theta_0} v + v \Delta_{\theta_0} v_t) dV_{\theta_0} \\
&= \int_{S^{2n+1}} (x, \bar{x}) \alpha df_\phi \cdot \xi dV_h + 2\left(2 + \frac{2}{n}\right) \int_{S^{2n+1}} v_t \langle \nabla_{\theta_0} (x, \bar{x}), \nabla_{\theta_0} v \rangle_{\theta_0} dV_{\theta_0} \\
&\quad + \left(2 + \frac{2}{n}\right) \int_{S^{2n+1}} (x, \bar{x}) v_t \left[\alpha f_\phi v^{1+\frac{2}{n}} - \left(\frac{nR_{\theta_0}}{n+1} + \frac{n}{2} \right) v + 2\Delta_{\theta_0} v \right] dV_{\theta_0} \\
&= \left[\int_{S^{2n+1}} (x, \bar{x}) \alpha df_\phi \cdot \xi dV_h + \left(2 + \frac{2}{n}\right) \int_{S^{2n+1}} (x, \bar{x}) v_t (\alpha f_\phi v^{1+\frac{2}{n}} - R_{\theta_0} v) dV_{\theta_0} \right] \\
&\quad + 2\left(2 + \frac{2}{n}\right) \int_{S^{2n+1}} (x, \bar{x}) v_t \Delta_{\theta_0} v dV_{\theta_0} + 2\left(2 + \frac{2}{n}\right) \int_{S^{2n+1}} v_t \langle \nabla_{\theta_0} (x, \bar{x}), \nabla_{\theta_0} v \rangle_{\theta_0} dV_{\theta_0} \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

By (3.2), Lemma 2.3, Lemma 3.1 and Lemma 4.1, by integration by parts and Hölder's inequality, we obtain

$$\begin{aligned}
I_1 &= \int_{S^{2n+1}} (x, \bar{x}) \left[(n+1)(\alpha f_\phi - R_h) + v^{-(2+\frac{2}{n})} \operatorname{div}'_{\theta_0} (v^{2+\frac{2}{n}} \xi) \right] (\alpha f_\phi v^{2+\frac{2}{n}} - R_{\theta_0} v^2) dV_{\theta_0} \\
&\quad + \int_{S^{2n+1}} (x, \bar{x}) \alpha df_\phi \cdot \xi dV_h \\
&= (n+1) \int_{S^{2n+1}} (x, \bar{x}) (\alpha f_\phi - R_h) (\alpha f_\phi v^{2+\frac{2}{n}} - R_{\theta_0} v^2) dV_{\theta_0} \\
&\quad - \int_{S^{2n+1}} (\xi, \vec{0}) (\alpha f_\phi v^{2+\frac{2}{n}} - R_{\theta_0} v^2) dV_{\theta_0} - \frac{2}{n} R_{\theta_0} \int_{S^{2n+1}} (x, \bar{x}) v (dv \cdot \xi) dV_{\theta_0} \\
&\leq C \|\alpha f_\phi v^{2+\frac{2}{n}} - R_{\theta_0} v^2\|_{L^2(S^{2n+1}, \theta_0)} (\|\alpha f_\phi - R_h\|_{L^2(S^{2n+1}, h)} + \|\xi\|_{L^\infty}) \\
&\quad + C \|\nabla_{\theta_0} v\|_{L^2(S^{2n+1}, \theta_0)} \|\xi\|_{L^\infty} = o(1) F_2(t)^{\frac{1}{2}},
\end{aligned}$$

where $\operatorname{div}'_{\theta_0}$ is the subdivergence operator of type $(1, 0)$ with respect to the contact form θ_0 (see [8] for the definition). By (3.2), Lemma 2.3 and Lemma 3.1, we get

$$\begin{aligned}
I_2 &= (2n+2) \int_{S^{2n+1}} (x, \bar{x}) (\alpha f_\phi - R_h) v \Delta_{\theta_0} v dV_{\theta_0} \\
&\quad + 2 \int_{S^{2n+1}} (x, \bar{x}) v^{-(1+\frac{2}{n})} \operatorname{div}'_{\theta_0} (v^{2+\frac{2}{n}} \xi) \Delta_{\theta_0} v dV_{\theta_0} \\
&= (2n+2) \int_{S^{2n+1}} (x, \bar{x}) (\alpha f_\phi - R_h) v \Delta_{\theta_0} v dV_{\theta_0} - 2 \int_{S^{2n+1}} (\xi, \vec{0}) v \Delta_{\theta_0} v dV_{\theta_0} \\
&\quad + 2(1 + \frac{2}{n}) \int_{S^{2n+1}} (x, \bar{x}) (dv \cdot \xi) \Delta_{\theta_0} v dV_{\theta_0} - 2 \int_{S^{2n+1}} (x, \bar{x}) v (d(\Delta_{\theta_0} v) \cdot \xi) dV_{\theta_0} \\
&\leq C (\|\Delta_{\theta_0} v\|_{L^2(S^{2n+1}, \theta_0)} F_2(t)^{\frac{1}{2}} + o(1) \|\xi\|_{L^\infty}) = o(1) F_2(t)^{\frac{1}{2}},
\end{aligned}$$

where we have used the estimate

$$\begin{aligned}
&- 2 \int_{S^{2n+1}} (x, \bar{x}) v (d(\Delta_{\theta_0} v) \cdot \xi) dV_{\theta_0} = 2 \int_{S^{2n+1}} (x, \bar{x}) v [d(R_h v^{1+\frac{2}{n}} - R_{\theta_0} v) \cdot \xi] dV_{\theta_0} \\
&= 2 \int_{S^{2n+1}} (x, \bar{x}) v [d((R_h - \alpha f_\phi) v^{1+\frac{2}{n}} + (\alpha f_\phi - R_{\theta_0}) v^{1+\frac{2}{n}} + (v^{1+\frac{2}{n}} - v) R_{\theta_0}) \cdot \xi] dV_{\theta_0} \\
&= o(1) \|\xi\|_{L^\infty},
\end{aligned}$$

thanks to Lemma 2.3, Lemma 4.1 and Lemma 4.3. Similarly, we find that

$$\begin{aligned}
I_3 &= (2n+2) \int_{S^{2n+1}} (\alpha f_\phi - R_h) v \langle \nabla_{\theta_0} (x, \bar{x}), \nabla_{\theta_0} v \rangle_{\theta_0} dV_{\theta_0} \\
&\quad + 2 \int_{S^{2n+1}} v^{-(1+\frac{2}{n})} \operatorname{div}_{\theta_0} (v^{2+\frac{2}{n}} \xi) \langle \nabla_{\theta_0} (x, \bar{x}), \nabla_{\theta_0} v \rangle_{\theta_0} dV_{\theta_0} \\
&= (2n+2) \int_{S^{2n+1}} (\alpha f_\phi - R_h) v \langle \nabla_{\theta_0} (x, \bar{x}), \nabla_{\theta_0} v \rangle_{\theta_0} dV_{\theta_0} \\
&\quad + 2(1 + \frac{2}{n}) \int_{S^{2n+1}} (dv \cdot \xi) \langle \nabla_{\theta_0} (x, \bar{x}), \nabla_{\theta_0} v \rangle_{\theta_0} dV_{\theta_0} \\
&\quad - 2 \int_{S^{2n+1}} v (d(\langle \nabla_{\theta_0} (x, \bar{x}), \nabla_{\theta_0} v \rangle_{\theta_0}) \cdot \xi) dV_{\theta_0} \\
&\leq C (\|\nabla_{\theta_0} v\|_{L^2(S^{2n+1}, \theta_0)} F_2(t)^{\frac{1}{2}} + \|\xi\|_{L^\infty} \|v - 1\|_{S^2_1(S^{2n+1}, \theta_0)}) = o(1) F_2(t)^{\frac{1}{2}}.
\end{aligned}$$

Inserting these estimates of I_1 , I_2 and I_3 into (4.8), we obtain the desired result. \square

Lemma 4.6. *For sufficiently large time t , there holds*

$$F_2(t) = (1 + o(1))|B(t)|^2$$

with error $o(1) \rightarrow 0$ as $t \rightarrow \infty$.

Proof. For brevity, we set $\widehat{F}_2(t) = \sum_{i=2n+2}^{\infty} |\beta_{\theta}^i|^2$. By (4.6), we have

$$\begin{aligned} F_2(t) &= \int_{S^{2n+1}} (\alpha f - R_{\theta})^2 dV_{\theta} = \sum_{i,j=1}^{\infty} \beta_{\theta}^i \beta_{\theta}^j \int_{S^{2n+1}} \varphi_i^{\theta} \varphi_j^{\theta} dV_{\theta} \\ (4.9) \quad &= \sum_{i=1}^{\infty} |\beta_{\theta}^i|^2 = |\beta_{\theta}|^2 + \widehat{F}_2(t) = |B|^2 + \widehat{F}_2(t) + o(1)F_2(t). \end{aligned}$$

Since

$$\begin{aligned} G_2(t) &= \int_{S^{2n+1}} |\nabla_{\theta}(\alpha f - R_{\theta})|_{\theta}^2 dV_{\theta} = - \int_{S^{2n+1}} (\alpha f - R_{\theta}) \Delta_{\theta}(\alpha f - R_{\theta}) dV_{\theta} \\ &= \sum_{i,j=1}^{\infty} \beta_{\theta}^i \beta_{\theta}^j \int_{S^{2n+1}} \varphi_i^{\theta} (-\Delta_{\theta} \varphi_j^{\theta}) dV_{\theta} \\ &= \sum_{i,j=1}^{\infty} \beta_{\theta}^i \beta_{\theta}^j \lambda_j^{\theta} \int_{S^{2n+1}} \varphi_i^{\theta} \varphi_j^{\theta} dV_{\theta} = \sum_{i=1}^{\infty} \lambda_i^{\theta} |\beta_{\theta}^i|^2, \end{aligned}$$

we have

$$\begin{aligned} \frac{n}{2} F_2(t) - G_2(t) &= \frac{n}{2} \sum_{i=1}^{\infty} |\beta_{\theta}^i|^2 - \sum_{i=1}^{\infty} \lambda_i^{\theta} |\beta_{\theta}^i|^2 \\ (4.10) \quad &= \frac{n}{2} \sum_{i=1}^{\infty} |\beta_{\theta}^i|^2 - \sum_{i=1}^{\infty} \lambda_i |\beta_{\theta}^i|^2 + \sum_{i=1}^{\infty} (\lambda_i - \lambda_i^{\theta}) |\beta_{\theta}^i|^2 \\ &= \sum_{i=2n+3}^{\infty} \left(\frac{n}{2} - \lambda_i \right) |\beta_{\theta}^i|^2 + o(1) \sum_{i=1}^{\infty} |\beta_{\theta}^i|^2 \\ &\leq \left(\frac{n}{2} - \lambda_{2n+3} \right) \widehat{F}_2(t) + o(1) F_2(t), \end{aligned}$$

where we have used Lemma 4.4 and the fact that $0 = \lambda_0 < \lambda_1 = \dots \lambda_{2n+2} = \frac{n}{2} < \lambda_{2n+3} \leq \lambda_i$ for $i \geq 2n+3$. From (4.10) and Lemma 4.2, we deduce

$$\begin{aligned} (4.11) \quad \frac{d}{dt} F_2(t) &\leq (n+1+o(1))(nF_2(t) - 2G_2(t)) + o(1)F_2(t) \\ &\leq 2(n+1) \left(\frac{n}{2} - \lambda_{2n+3} \right) \widehat{F}_2(t) + o(1)F_2(t). \end{aligned}$$

Suppose there exists some sufficiently large time t_1 such that $|B(t_1)|^2 \geq \widehat{F}_2(t_1)$. Denote

$$F_2(t) = (1 + \delta(t))|B(t)|^2$$

near t_1 . Then we have $-\frac{1}{2} \leq \delta(t) \leq 2$ for all time t sufficiently close to t_1 by continuity of $\frac{\widehat{F}_2(t)}{|B(t)|^2}$ at $t = t_1$. By (4.11), we get

$$\begin{aligned}
 (4.12) \quad & \frac{d\delta(t)}{dt} |B(t)|^2 + 2(1 + \delta(t))B(t) \frac{dB(t)}{dt} \\
 &= \frac{d}{dt} F_2(t) \leq 2(n+1) \left(\frac{n}{2} - \lambda_{2n+3} \right) \widehat{F}_2(t) + o(1) F_2(t) \\
 &= 2(n+1) \left(\frac{n}{2} - \lambda_{2n+3} \right) \delta(t) |B(t)|^2 + o(1) F_2(t).
 \end{aligned}$$

It follows from Lemma 4.5 that

$$\left| B(t) \frac{dB(t)}{dt} \right| = o(1) |B(t)| F_2(t)^{\frac{1}{2}} \leq o(1) F_2(t)$$

since $|B(t)| \leq F_2(t)$. Substituting it into (4.12) and dividing $|B(t)|^2$ on both sides, we find

$$\frac{d\delta(t)}{dt} \leq 2(n+1) \left(\frac{n}{2} - \lambda_{2n+3} \right) \delta(t) + o(1) \frac{F_2(t)}{|B(t)|^2} = \left[2(n+1) \left(\frac{n}{2} - \lambda_{2n+3} \right) + o(1) \right] \delta(t).$$

Since $\lambda_{2n+3} > \frac{n}{2}$, this implies $\delta(t) \rightarrow 0$ as $t \rightarrow \infty$, as required. It follows from this argument that our choice of t_1 must satisfies that $|o(1)| \leq (n+1)(\lambda_{2n+3} - \frac{n}{2})$ when $t \geq t_1$.

It reduces to seek a time t_1 such that $|B(t_1)|^2 \geq \widehat{F}_2(t_1)$ for sufficiently large t_1 . Assume, on the contrary, that $|B(t)|^2 < \widehat{F}_2(t)$ for all sufficiently large t . Therefore by (4.9)

$$F_2(t) = |B|^2 + \widehat{F}_2(t) + o(1) F_2(t) < 2\widehat{F}_2(t) + o(1) F_2(t),$$

which implies that

$$\begin{aligned}
 \frac{d}{dt} F_2(t) &\leq -2(n+1) \left(\lambda_{2n+3} - \frac{n}{2} \right) \widehat{F}_2(t) + o(1) F_2(t) \\
 &\leq -(n+1) \left(\lambda_{2n+3} - \frac{n}{2} \right) F_2(t) + o(1) F_2(t)
 \end{aligned}$$

by (4.11). Hence, we have

$$(4.13) \quad F_2(t) \leq C e^{-\frac{(n+1)}{2} (\lambda_{2n+3} - \frac{n}{2}) t}$$

for $t \geq t_2$ and C depending only on t_2 . Let Q be the unique concentration point described in Theorem 2.2, and $B_{r_0}(Q) = B_{r_0}(Q, \theta_0)$. For any $r_0 > 0$, we have

$$\begin{aligned}
 \left| \frac{d}{dt} \text{Vol}(B_{r_0}(Q), \theta) \right| &= \left| \frac{d}{dt} \left(\int_{B_{r_0}(Q)} dV_\theta \right) \right| = (n+1) \left| \int_{B_{r_0}(Q)} (\alpha f - R_\theta) dV_\theta \right| \\
 &\leq (n+1) \text{Vol}(S^{2n+1}, \theta)^{\frac{1}{2}} \left(\int_{B_{r_0}(Q)} (\alpha f - R_\theta)^2 dV_\theta \right)^{\frac{1}{2}} \\
 &\leq (n+1) \text{Vol}(S^{2n+1}, \theta_0)^{\frac{1}{2}} F_2(t)^{\frac{1}{2}} \\
 &\leq C e^{-\frac{(n+1)}{4} (\lambda_{2n+3} - \frac{n}{2}) t} \quad \text{for } t \geq t_2,
 \end{aligned}$$

by (2.4) and (4.13). Thus, by integrating the above inequality from t_2 to a larger t , we get

(4.14)

$$\begin{aligned} \text{Vol}(B_{r_0}(Q), \theta(t)) &< \text{Vol}(B_{r_0}(Q), \theta(t_2)) + \frac{4C}{(n+1)(\lambda_{2n+3} - \frac{n}{2})} e^{-\frac{(n+1)}{4}(\lambda_{2n+3} - \frac{n}{2})t_2} \\ &< \text{Vol}(S^{2n+1}, \theta_0)/2 \end{aligned}$$

uniformly for $t \geq t_2$ by first choosing t_2 sufficiently large and then choosing r_0 sufficiently small. On the other hand, from Theorem 2.2, we know that

$$\text{Vol}(B_{r_0}(Q), \theta(t)) \rightarrow \text{Vol}(S^{2n+1}, \theta_0) \quad \text{as } t \rightarrow \infty$$

which yields a contradiction with (4.14). Thus the proof is complete. \square

Lemma 4.7. *With a uniform constant $C > 0$, there holds*

$$\|v - 1\|_{S_2^2(S^{2n+1}, \theta_0)} \leq C(F_2(t)^{\frac{1}{2}} + \|f_\phi - f(\Theta(t))\|_{L^2(S^{2n+1}, \theta_0)}).$$

Proof. Expand $v^{2+\frac{2}{n}} - 1$ and $v - 1$ in terms of eigenfunctions to get

$$v^{2+\frac{2}{n}} - 1 = \sum_{i=0}^{\infty} V^i \varphi_i \quad \text{and} \quad v - 1 = \sum_{i=0}^{\infty} v^i \varphi_i.$$

By Proposition 2.1 in [17], we have

$$\int_{S^{2n+1}} (v^{2+\frac{2}{n}} - 1) dV_{\theta_0} = \int_{S^{2n+1}} (u^{2+\frac{2}{n}} - 1) dV_{\theta_0} = 0$$

which implies that $V^0 = 0$. On the other hand, due to the normalization (2.9) of v , we have $V^i = 0$ for $1 \leq i \leq 2n+2$. Observe that by Taylor's expansion and Lemma 2.3,

$$\begin{aligned} (2 + \frac{2}{n})v^i &= (2 + \frac{2}{n}) \int_{S^{2n+1}} (v - 1) \varphi_i dV_{\theta_0} \\ &= \int_{S^{2n+1}} (v^{2+\frac{2}{n}} - 1) \varphi_i dV_{\theta_0} + O(\|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)}^2) \\ &= V^i + o(1)\|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)}. \end{aligned}$$

Thus it follows that

$$(4.15) \quad \sum_{i=0}^{2n+2} |v^i|^2 = o(1)\|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)}^2.$$

We may rewrite (2.13) in the form

(4.16)

$$\begin{aligned} -(2 + \frac{2}{n})\Delta_{\theta_0} v &= (R_h v^{1+\frac{2}{n}} - R_{\theta_0} v) \\ &= \left[(R_h - \alpha f_\phi) + (\alpha f_\phi - \alpha f(\Theta(t))) \right. \\ &\quad \left. + \left(\alpha f(\Theta(t)) - \frac{1}{\text{Vol}(S^{2n+1}, \theta_0)} \int_{S^{2n+1}} R_h dV_h \right) \right] v^{1+\frac{2}{n}} \\ &\quad + \left(\frac{1}{\text{Vol}(S^{2n+1}, \theta_0)} \int_{S^{2n+1}} R_h dV_h - R_{\theta_0} \right) v^{1+\frac{2}{n}} + R_{\theta_0} (v^{1+\frac{2}{n}} - v). \end{aligned}$$

We are going to estimate the terms on the right hand side of (4.16). By (4.34), Proposition 2.1 and Lemma 2.4 in [17], and by Lemma 2.3, we have

$$\begin{aligned}
 (4.17) \quad & \left| \alpha f(\Theta(t)) \text{Vol}(S^{2n+1}, \theta_0) - \int_{S^{2n+1}} R_h dV_h \right| \\
 & \leq \left| \alpha \int_{S^{2n+1}} (f(\Theta(t)) - f_\phi) dV_h \right| + \left| \int_{S^{2n+1}} (\alpha f - R_h) dV_h \right| \\
 & \leq C \|f(\Theta(t)) - f_\phi\|_{L^2(S^{2n+1}, \theta_0)} + \text{Vol}(S^{2n+1}, \theta_0)^{\frac{1}{2}} \left(\int_{S^{2n+1}} (\alpha f - R_h)^2 dV_h \right)^{\frac{1}{2}}.
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 E(v-1) &= \int_{S^{2n+1}} \left(\left(2 + \frac{2}{n}\right) |\nabla_{\theta_0} v|_{\theta_0}^2 + R_{\theta_0} (v-1)^2 \right) dV_{\theta_0} \\
 &= \int_{S^{2n+1}} \left(\left(2 + \frac{2}{n}\right) |\nabla_{\theta_0} v|_{\theta_0}^2 + R_{\theta_0} v^2 \right) dV_{\theta_0} - 2R_{\theta_0} \int_{S^{2n+1}} (v-1) dV_{\theta_0} \\
 &\quad - R_{\theta_0} \int_{S^{2n+1}} dV_{\theta_0} \\
 &= \int_{S^{2n+1}} R_h dV_h - 2R_{\theta_0} \int_{S^{2n+1}} (v-1) dV_{\theta_0} - R_{\theta_0} \text{Vol}(S^{2n+1}, \theta_0),
 \end{aligned}$$

which implies that

$$\begin{aligned}
 (4.18) \quad & \left| \int_{S^{2n+1}} R_h dV_h - R_{\theta_0} \text{Vol}(S^{2n+1}, \theta_0) \right| \leq E(v-1) + 2R_{\theta_0} \left| \int_{S^{2n+1}} (v-1) dV_{\theta_0} \right| \\
 & = E(v-1) + C|v^0| \\
 & \leq E(v-1) + o(1) \|v-1\|_{S_1^2(S^{2n+1}, \theta_0)}
 \end{aligned}$$

by (4.15). Observe that we have

$$\begin{aligned}
 (4.19) \quad E(v-1) &= \left(2 + \frac{2}{n}\right) \int_{S^{2n+1}} |\nabla_{\theta_0} v|_{\theta_0}^2 dV_{\theta_0} \\
 &\quad + R_{\theta_0} \int_{S^{2n+1}} \left(v - \frac{1}{\text{Vol}(S^{2n+1}, \theta_0)} \int_{S^{2n+1}} v dV_{\theta_0} \right)^2 dV_{\theta_0} \\
 &\quad + \frac{R_{\theta_0}}{\text{Vol}(S^{2n+1}, \theta_0)} \left(\int_{S^{2n+1}} (v-1) dV_{\theta_0} \right)^2.
 \end{aligned}$$

Since the first eigenvalue of the sub-Laplacian of θ_0 is $n/2$, together with (4.15), we obtain from (4.19) that

$$\begin{aligned}
 (4.20) \quad E(v-1) &\leq \left(2 + \frac{2}{n}\right) \int_{S^{2n+1}} |\nabla_{\theta_0} v|_{\theta_0}^2 dV_{\theta_0} + \frac{2R_{\theta_0}}{n} \int_{S^{2n+1}} |\nabla_{\theta_0} v|_{\theta_0}^2 dV_{\theta_0} \\
 &\quad + o(1) \|v-1\|_{S_1^2(S^{2n+1}, \theta_0)}^2 \\
 &= \left(1 + \frac{2}{n}\right) (n+1) \int_{S^{2n+1}} |\nabla_{\theta_0} v|_{\theta_0}^2 dV_{\theta_0} + o(1) \|v-1\|_{S_1^2(S^{2n+1}, \theta_0)}^2.
 \end{aligned}$$

We also need the following:

$$(4.21) \quad v^{1+\frac{2}{n}} - v = \frac{2}{n} (v-1) + o(|v-1|).$$

Since $\lambda_{2n+3} \geq (1 - 2\epsilon(n))\frac{n}{2}$ for some constant $\epsilon(n) > 0$ depending only n , with sufficiently small $\delta > 0$ and $\epsilon_0 > 0$ we have

$$(4.22) \quad (2 + \frac{2}{n})^{-2} R_{\theta_0}^2 (\frac{2}{n})^2 (1 + \delta)(1 + \epsilon_0) = \frac{n^2}{4} (1 + \delta)(1 + \epsilon_0) \leq (1 - \epsilon(n)) \lambda_{2n+3}^2,$$

we find that

$$(4.23) \quad \begin{aligned} \sum_{i=0}^{\infty} \lambda_i^2 |v^i|^2 &= \|\Delta_{\theta_0} v\|_{L^2(S^{2n+1}, \theta_0)}^2 \\ &\leq C(\delta) \left[\|(R_h - \alpha f_\phi) v^{1+\frac{2}{n}}\|_{L^2(S^{2n+1}, \theta_0)}^2 + \|(\alpha f_\phi - \alpha f(\Theta(t))) v^{1+\frac{2}{n}}\|_{L^2(S^{2n+1}, \theta_0)}^2 \right. \\ &\quad \left. + \left(\alpha f(\Theta(t)) - \frac{1}{\text{Vol}(S^{2n+1}, \theta_0)} \int_{S^{2n+1}} R_h dV_h \right)^2 \|v^{1+\frac{2}{n}}\|_{L^2(S^{2n+1}, \theta_0)}^2 \right] \\ &\quad + (2 + \frac{2}{n})^{-2} (1 + \delta) \left[(1 + \epsilon_0) R_{\theta_0}^2 \|v^{1+\frac{2}{n}} - v\|_{L^2(S^{2n+1}, \theta_0)}^2 \right. \\ &\quad \left. + (1 + \epsilon_0^{-1}) \left(\frac{1}{\text{Vol}(S^{2n+1}, \theta_0)} \int_{S^{2n+1}} R_h dV_h - R_{\theta_0} \right)^2 \|v^{1+\frac{2}{n}}\|_{L^2(S^{2n+1}, \theta_0)}^2 \right] \\ &\leq C(\delta) \left(F_2(t) + \|f_\phi - f(\Theta(t))\|_{L^2(S^{2n+1}, \theta_0)}^2 \right) \\ &\quad + (2 + \frac{2}{n})^{-2} R_{\theta_0}^2 (\frac{2}{n})^2 (1 + \delta)(1 + \epsilon_0) \|v - 1\|_{L^2(S^{2n+1}, \theta_0)}^2 + o(1) \|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)}^2 \\ &\quad + 2(2 + \frac{2}{n})^{-2} (1 + \frac{2}{n})^2 (n+1)^2 \cdot \frac{(1 + \delta)(1 + \epsilon_0^{-1})}{\text{Vol}(S^{2n+1}, \theta_0)^2} \left(\int_{S^{2n+1}} |\nabla_{\theta_0} v|_{\theta_0}^2 dV_{\theta_0} \right)^2 \\ &\leq C(\delta) \left(F_2(t) + \|f_\phi - f(\Theta(t))\|_{L^2(S^{2n+1}, \theta_0)}^2 \right) + (1 - \epsilon(n)) \lambda_{2n+3}^2 \sum_{i=0}^{\infty} |v^i|^2 + o(1) \|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)}^2 \\ &\quad + \frac{(n+2)^2}{2} \cdot \frac{(1 + \delta)(1 + \epsilon_0^{-1})}{\text{Vol}(S^{2n+1}, \theta_0)^2} \left(\int_{S^{2n+1}} |\Delta_{\theta_0} v|^2 dV_{\theta_0} \right) \left(\int_{S^{2n+1}} (v-1)^2 dV_{\theta_0} \right) \end{aligned}$$

where the first inequality follows from (4.16) and Young's inequality, and the second inequality follows from (4.17), (4.18), (4.20), (4.21), and Lemma 2.3, and the last inequality follows from (4.22) and

$$\begin{aligned} \left(\int_{S^{2n+1}} |\nabla_{\theta_0} v|_{\theta_0}^2 dV_{\theta_0} \right)^2 &= \left(\int_{S^{2n+1}} (v-1) \Delta_{\theta_0} v dV_{\theta_0} \right)^2 \\ &\leq \left(\int_{S^{2n+1}} |\Delta_{\theta_0} v|^2 dV_{\theta_0} \right) \left(\int_{S^{2n+1}} (v-1)^2 dV_{\theta_0} \right) \end{aligned}$$

by Hölder's inequality. Since $\int_{S^{2n+1}} (v-1)^2 dV_{\theta_0} \rightarrow 0$ as $t \rightarrow \infty$, we can choose sufficiently large t_0 such that if $t \geq t_0$, then

$$(4.24) \quad \frac{(n+2)^2}{2} \cdot \frac{(1 + \delta)(1 + \epsilon_0^{-1})}{\text{Vol}(S^{2n+1}, \theta_0)} \left(\int_{S^{2n+1}} (v-1)^2 dV_{\theta_0} \right) < \frac{1}{2}.$$

Thus by (4.15) and (4.24), we can absorb the last three terms on the right hand side of (4.23) to conclude that

$$(4.25) \quad \int_{S^{2n+1}} |\Delta_{\theta_0} v|^2 dV_{\theta_0} \leq C(\delta) \left(F_2(t) + \|f_\phi - f(\Theta(t))\|_{L^2(S^{2n+1}, \theta_0)}^2 \right) + o(1) \|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)}^2.$$

Now we also have

$$(4.26) \quad \begin{aligned} & \frac{1}{2} \int_{S^{2n+1}} |\nabla_{\theta_0} v|_{\theta_0}^2 dV_{\theta_0} + \int_{S^{2n+1}} (v - 1)^2 dV_{\theta_0} \\ &= \frac{1}{2} \int_{S^{2n+1}} |\nabla_{\theta_0} v|_{\theta_0}^2 dV_{\theta_0} + \int_{S^{2n+1}} \left(v - \frac{1}{\text{Vol}(S^{2n+1}, \theta_0)} \int_{S^{2n+1}} v dV_{\theta_0} \right)^2 dV_{\theta_0} \\ & \quad + \frac{1}{\text{Vol}(S^{2n+1}, \theta_0)} \left(\int_{S^{2n+1}} (v - 1) dV_{\theta_0} \right)^2 \\ &\leq \left(\frac{1}{2} + \frac{2}{n} \right) \int_{S^{2n+1}} |\nabla_{\theta_0} v|_{\theta_0}^2 dV_{\theta_0} + o(1) \|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)}^2 \\ &= \left(\frac{1}{2} + \frac{2}{n} \right) \int_{S^{2n+1}} (v - 1) \Delta_{\theta_0} v dV_{\theta_0} + o(1) \|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)}^2 \\ &\leq \frac{1}{2} \left(\frac{1}{2} + \frac{2}{n} \right)^2 \int_{S^{2n+1}} |\Delta_{\theta_0} v|^2 dV_{\theta_0} + \frac{1}{2} \int_{S^{2n+1}} (v - 1)^2 dV_{\theta_0} + o(1) \|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)}^2 \end{aligned}$$

where the first inequality follows from (4.15) and the fact that the first eigenvalue λ_1 of the sub-Laplacian for θ_0 is $n/2$. By absorbing the second term on the right hand side of (4.26) to the left hand side, we get

$$\begin{aligned} & \frac{1}{2} \int_{S^{2n+1}} (v - 1)^2 dV_{\theta_0} + \frac{1}{2} \int_{S^{2n+1}} |\nabla_{\theta_0} v|_{\theta_0}^2 dV_{\theta_0} \\ &\leq \frac{1}{2} \left(\frac{1}{2} + \frac{2}{n} \right)^2 \int_{S^{2n+1}} |\Delta_{\theta_0} v|^2 dV_{\theta_0} + o(1) \|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)}^2 \\ &\leq C(\delta) \left(F_2(t) + \|f_\phi - f(\Theta(t))\|_{L^2(S^{2n+1}, \theta_0)}^2 \right) + o(1) \|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)}^2 \end{aligned}$$

by (4.25). Hence we conclude that

$$(4.27) \quad \int_{S^{2n+1}} (v - 1)^2 dV_{\theta_0} + \int_{S^{2n+1}} |\nabla_{\theta_0} v|_{\theta_0}^2 dV_{\theta_0} \leq C(\delta) \left(F_2(t) + \|f_\phi - f(\Theta(t))\|_{L^2(S^{2n+1}, \theta_0)}^2 \right).$$

Substituting (4.27) back to (4.25), we obtain

$$(4.28) \quad \int_{S^{2n+1}} |\Delta_{\theta_0} v|^2 dV_{\theta_0} \leq C(\delta) \left(F_2(t) + \|f_\phi - f(\Theta(t))\|_{L^2(S^{2n+1}, \theta_0)}^2 \right).$$

Now the assertion follows from (4.27) and (4.28). \square

Lemma 4.8. *For all $t > 0$, there hold*

$$\begin{aligned} & b - \langle b, (\widehat{\Theta(t)}, \widehat{\Theta(t)}) \rangle (\widehat{\Theta(t)}, \widehat{\Theta(t)}) \\ &= \epsilon A_1 \left(\frac{\partial f(\widehat{\Theta(t)})}{\partial a_1} + \sqrt{-1} \frac{\partial f(\widehat{\Theta(t)})}{\partial b_1}, \dots, \frac{\partial f(\widehat{\Theta(t)})}{\partial a_n} + \sqrt{-1} \frac{\partial f(\widehat{\Theta(t)})}{\partial b_n}, 0, \right. \\ & \quad \left. \frac{\partial f(\widehat{\Theta(t)})}{\partial a_1} - \sqrt{-1} \frac{\partial f(\widehat{\Theta(t)})}{\partial b_1}, \dots, \frac{\partial f(\widehat{\Theta(t)})}{\partial a_n} - \sqrt{-1} \frac{\partial f(\widehat{\Theta(t)})}{\partial b_n}, 0 \right) + O(\epsilon^2) \end{aligned}$$

and

$$\langle b, (\widehat{\Theta(t)}, \widehat{\Theta(t)}) \rangle = -2\epsilon^2 A_2 \alpha \Delta_{\theta_0} f(\widehat{\Theta(t)}) + O(\epsilon^2) |\nabla_{\theta_0} f(\widehat{\Theta(t)})|_{\theta_0}^2 + O(\epsilon^3),$$

where A_1 and A_2 are positive constants defined as in (4.38) and (4.46) respectively.

Proof. Using (3.1), we find

$$\begin{aligned} \frac{n}{2}b &= \frac{n}{2} \int_{S^{2n+1}} (x, \bar{x})(\alpha f_\phi - R_h) dV_h = - \int_{S^{2n+1}} \Delta_{\theta_0}(x, \bar{x})(\alpha f_\phi - R_h) dV_h \\ &= \alpha \int_{S^{2n+1}} \langle \nabla_{\theta_0}(x, \bar{x}), \nabla_{\theta_0} f_\phi \rangle_{\theta_0} dV_h + E_1 \end{aligned}$$

with error

$$E_1 = (2 + \frac{2}{n}) \int_{S^{2n+1}} (\alpha f_\phi - R_h) \langle \nabla_{\theta_0}(x, \bar{x}), \nabla_{\theta_0} v \rangle_{\theta_0} v^{1+\frac{2}{n}} dV_{\theta_0}.$$

By Hölder's inequality and Lemma 2.3, the error term can be estimated as (4.29)

$$|E_1| \leq C \|\nabla_{\theta_0} v\|_{L^2(S^{2n+1}, \theta_0)} \|\alpha f_\phi - R_h\|_{L^2(S^{2n+1}, h)} \leq C \|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)} F_2^{\frac{1}{2}}.$$

Thus we obtain

$$\begin{aligned} \frac{n}{2}b &= \alpha \int_{S^{2n+1}} \langle \nabla_{\theta_0}(x, \bar{x}), \nabla_{\theta_0} f_\phi \rangle_{\theta_0} dV_{\theta_0} + E_1 + E_2 \\ (4.30) \quad &= \alpha \int_{S^{2n+1}} \Delta_{\theta_0}(x, \bar{x})(f_\phi - f(\widehat{\Theta(t)})) dV_{\theta_0} + E_1 + E_2 \\ &= \frac{n}{2} \alpha \int_{S^{2n+1}} (x, \bar{x})(f_\phi - f(\widehat{\Theta(t)})) dV_{\theta_0} + E_1 + E_2, \end{aligned}$$

where

$$\begin{aligned} E_2 &= \alpha \int_{S^{2n+1}} \langle \nabla_{\theta_0}(x, \bar{x}), \nabla_{\theta_0} f_\phi \rangle_{\theta_0} (v^{2+\frac{2}{n}} - 1) dV_{\theta_0} \\ &= \frac{n}{2} \alpha \int_{S^{2n+1}} (x, \bar{x})(f_\phi - f(\widehat{\Theta(t)})) (v^{2+\frac{2}{n}} - 1) dV_{\theta_0} \\ &\quad - (2 + \frac{2}{n}) \alpha \int_{S^{2n+1}} \langle \nabla_{\theta_0}(x, \bar{x}), \nabla_{\theta_0} v \rangle_{\theta_0} (f_\phi - f(\widehat{\Theta(t)})) v^{1+\frac{2}{n}} dV_{\theta_0}. \end{aligned}$$

Then it follows from Hölder's inequality and Lemma 2.3 that

$$\begin{aligned} (4.31) \quad |E_2| &\leq C(\|v - 1\|_{L^2(S^{2n+1}, \theta_0)} + \|\nabla_{\theta_0} v\|_{L^2(S^{2n+1}, \theta_0)}) \|f_\phi - f(\widehat{\Theta(t)})\|_{L^2(S^{2n+1}, \theta_0)} \\ &\leq \|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)} \|f_\phi - f(\widehat{\Theta(t)})\|_{L^2(S^{2n+1}, \theta_0)}. \end{aligned}$$

We henceforth focus on the term

$$\int_{S^{2n+1}} (x, \bar{x})(f_\phi - f(\widehat{\Theta(t)})) dV_{\theta_0}.$$

We will keep the coordinate for \mathbb{H}^n such that the north pole N of S^{2n+1} is $\widehat{\Theta(t)}$. If we use the tangent plane of the sphere at the north pole $N = (0, \dots, 0, 1)$ as local coordinates for S^{2n+1} , then

$$\phi(z, \tau) = \left(\frac{2\epsilon z}{1 + \epsilon^2 |z|^2 - \sqrt{-1}\epsilon^2 \tau}, \frac{1 - \epsilon^2 |z|^2 + \sqrt{-1}\epsilon^2 \tau}{1 + \epsilon^2 |z|^2 - \sqrt{-1}\epsilon^2 \tau} \right), \quad (z, \tau) \in \mathbb{H}^n$$

where $\epsilon = \frac{1}{r(t)}$. Hence, in $B_{\epsilon^{-1}}(0)$, we can expand around $(z, \tau) = (0, 0)$

$$\begin{aligned}
(4.32) \quad & (f_\phi - f(\widehat{\Theta(t)}))(\Psi(z, \tau)) \\
&= \sum_{i=1}^n \frac{\partial f(\phi(z, \tau))}{\partial a_i} \Big|_{(z, \tau)=(0,0)} a_i + \sum_{i=1}^n \frac{\partial f(\phi(z, \tau))}{\partial b_i} \Big|_{(z, \tau)=(0,0)} b_i \\
&\quad + \frac{\partial f(\phi(z, \tau))}{\partial \tau} \Big|_{(z, \tau)=(0,0)} \tau + \frac{1}{2} \frac{\partial^2 f(\phi(z, \tau))}{\partial \tau^2} \Big|_{(z, \tau)=(0,0)} \tau^2 \\
&\quad + \frac{1}{2} \sum_{i,j=1}^n \left(\frac{\partial^2 f(\phi(z, \tau))}{\partial a_i \partial a_j} \Big|_{(z, \tau)=(0,0)} a_i a_j + \frac{\partial^2 f(\phi(z, \tau))}{\partial b_i \partial b_j} \Big|_{(z, \tau)=(0,0)} b_i b_j \right) \\
&\quad + \frac{1}{2} \sum_{i=1}^n \left(\frac{\partial^2 f(\phi(z, \tau))}{\partial a_i \partial \tau} \Big|_{(z, \tau)=(0,0)} a_i \tau + \frac{\partial^2 f(\phi(z, \tau))}{\partial b_i \partial \tau} \Big|_{(z, \tau)=(0,0)} b_i \tau \right) \\
&\quad + O(\epsilon^3(|z|^4 + \tau^2)^{\frac{3}{4}}) \\
&= [df(\widehat{\Theta(t)}) \cdot d\phi|_{(z, \tau)=(0,0)}] \cdot (z, \tau) \\
&\quad + \frac{1}{2} (\nabla df)(\widehat{\Theta(t)})(d\phi|_{(z, \tau)=(0,0)}(z, \tau), d\phi|_{(z, \tau)=(0,0)}(z, \tau)) + O(\epsilon^3(|z|^4 + \tau^2)^{\frac{3}{4}}) \\
&= df(\widehat{\Theta(t)})(\epsilon z, \epsilon^2 \tau) + \frac{1}{2} (\nabla df)(\widehat{\Theta(t)})((\epsilon z, \epsilon^2 \tau), (\epsilon z, \epsilon^2 \tau)) + O(\epsilon^3(|z|^4 + \tau^2)^{\frac{3}{4}}),
\end{aligned}$$

where $z = (z_1, \dots, z_n) = (a_1 + \sqrt{-1}b_1, \dots, a_n + \sqrt{-1}b_n) \in \mathbb{C}^n$.

First, by (4.2) and the boundedness of x and f , we have

$$\begin{aligned}
(4.33) \quad & \left| \int_{\Psi(\mathbb{H}^n \setminus B_{\epsilon^{-1}}(0))} (x, \bar{x})(f_\phi - f(\widehat{\Theta(t)})) dV_{\theta_0} \right| \leq C \int_{\mathbb{H}^n \setminus B_{\epsilon^{-1}}(0)} \frac{dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} \\
&= O(\epsilon^{2n}).
\end{aligned}$$

Using (4.33), we can give a estimate of

$$\begin{aligned}
(4.34) \quad & \int_{S^{2n+1}} |f_\phi - f(\widehat{\Theta(t)})|^2 dV_{\theta_0} \\
&= \int_{B_{\epsilon^{-1}}(0)} |f(\Psi(z, \tau)) - f(\widehat{\Theta(t)})|^2 \frac{4^{n+1} dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} + O(\epsilon^{2n}) \\
&\leq C \epsilon^2 |\nabla_{\theta_0} f(\widehat{\Theta(t)})|_{\theta_0}^2 \int_{B_{\epsilon^{-1}}(0)} \frac{(\tau^2 + |z|^4)^{\frac{1}{2}} dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} \\
&\quad + C \epsilon^4 \int_{B_{\epsilon^{-1}}(0)} \frac{(\tau^2 + |z|^4) dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} + O(\epsilon^{2n}) \\
&\leq C |\nabla_{\theta_0} f(\widehat{\Theta(t)})|_{\theta_0}^2 \epsilon^2 + C \epsilon^3,
\end{aligned}$$

where we have used the estimate (4.1) in the last step. Next, by (4.32), we have

$$\begin{aligned}
(4.35) \quad & \int_{B_{\epsilon^{-1}}(0)} (x, \bar{x})(f(\Psi(z)) - f(\widehat{\Theta(t)})) \frac{4^{n+1} dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} \\
&= \int_{B_{\epsilon^{-1}}(0)} (x, \bar{x}) df(\widehat{\Theta(t)})(\epsilon z, \epsilon^2 \tau) \frac{4^{n+1} dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} \\
&\quad + \frac{1}{2} \int_{B_{\epsilon^{-1}}(0)} (x, \bar{x})(\nabla df)(\widehat{\Theta(t)})(\epsilon z, \epsilon^2 \tau), (\epsilon z, \epsilon^2 \tau) \frac{4^{n+1} dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} + E_3 \\
&:= (I_1, I_2) + (II_1, II_2) + E_3
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \int_{B_{\epsilon^{-1}}(0)} x df(\widehat{\Theta(t)})(\epsilon z, \epsilon^2 \tau) \frac{4^{n+1} dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}}, \\
I_2 &= \int_{B_{\epsilon^{-1}}(0)} \bar{x} df(\widehat{\Theta(t)})(\epsilon z, \epsilon^2 \tau) \frac{4^{n+1} dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}}, \\
II_1 &= \frac{1}{2} \int_{B_{\epsilon^{-1}}(0)} x (\nabla df)(\widehat{\Theta(t)})(\epsilon z, \epsilon^2 \tau), (\epsilon z, \epsilon^2 \tau) \frac{4^{n+1} dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}}, \\
II_2 &= \frac{1}{2} \int_{B_{\epsilon^{-1}}(0)} \bar{x} (\nabla df)(\widehat{\Theta(t)})(\epsilon z, \epsilon^2 \tau), (\epsilon z, \epsilon^2 \tau) \frac{4^{n+1} dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}},
\end{aligned}$$

with the error E_3 bounded by

$$(4.36) \quad |E_3| \leq C \epsilon^3 \int_{B_{\epsilon^{-1}}(0)} \frac{(|z|^4 + \tau^2)^{\frac{3}{4}}}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} dz d\tau = O(\epsilon^3)$$

by (4.1). Now we estimate b by dividing its components into two cases:

Case (i). We deal with the tangential part first. By (4.32) and symmetry, we have

$$\begin{aligned}
(4.37) \quad & I_1 - \langle I_1, \widehat{\Theta(t)} \rangle \widehat{\Theta(t)} \\
&= \int_{B_{\epsilon^{-1}}(0)} \left(\frac{2z}{1 + |z|^2 - \sqrt{-1}\tau}, 0 \right) df(\widehat{\Theta(t)})(\epsilon z, \epsilon^2 \tau) \frac{4^{n+1} dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} \\
&= \int_{B_{\epsilon^{-1}}(0)} (2z_1(1 + |z|^2 + \sqrt{-1}\tau), \dots, 2z_n(1 + |z|^2 + \sqrt{-1}\tau), 0) \\
&\quad \cdot \left(\sum_{i=1}^n \frac{\partial f(\widehat{\Theta(t)})}{\partial a_i} \epsilon a_i + \sum_{i=1}^n \frac{\partial f(\widehat{\Theta(t)})}{\partial b_i} \epsilon b_i + \frac{\partial f(\widehat{\Theta(t)})}{\partial \tau} \epsilon^2 \tau \right) \frac{4^{n+1} dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+2}} \\
&= 2\epsilon \int_{B_{\epsilon^{-1}}(0)} \left(\frac{\partial f(\widehat{\Theta(t)})}{\partial a_1} a_1^2 + \sqrt{-1} \frac{\partial f(\widehat{\Theta(t)})}{\partial b_1} b_1^2, \dots, \frac{\partial f(\widehat{\Theta(t)})}{\partial a_n} a_n^2 + \sqrt{-1} \frac{\partial f(\widehat{\Theta(t)})}{\partial b_n} b_n^2, 0 \right) \\
&\quad \cdot \frac{4^{n+1} (1 + |z|^2)}{(\tau^2 + (1 + |z|^2)^2)^{n+2}} dz d\tau \\
&= \epsilon A_1 \left(\frac{\partial f(\widehat{\Theta(t)})}{\partial a_1} + \sqrt{-1} \frac{\partial f(\widehat{\Theta(t)})}{\partial b_1}, \dots, \frac{\partial f(\widehat{\Theta(t)})}{\partial a_n} + \sqrt{-1} \frac{\partial f(\widehat{\Theta(t)})}{\partial b_n}, 0 \right) + O(\epsilon^{2n+1}),
\end{aligned}$$

where

$$(4.38) \quad A_1 = \frac{1}{n} \int_{\mathbb{H}^n} \frac{4^{n+1}|z|^2(1+|z|^2)dzd\tau}{(\tau^2 + (1+|z|^2)^2)^{n+2}} > 0,$$

since

$$\int_{\mathbb{H}^n \setminus B_{\epsilon^{-1}}(0)} \frac{4^{n+1}|z|^2(1+|z|^2)dzd\tau}{(\tau^2 + (1+|z|^2)^2)^{n+2}} \leq \int_{\mathbb{H}^n \setminus B_{\epsilon^{-1}}(0)} \frac{4^{n+1}dzd\tau}{(\tau^2 + (1+|z|^2)^2)^{n+1}} = O(\epsilon^{2n})$$

by (4.2). Similarly, we have

$$(4.39) \quad \begin{aligned} I_2 - \langle I_2, \widehat{\Theta(t)} \widehat{\Theta(t)} \rangle &= \int_{B_{\epsilon^{-1}}(0)} \left(\frac{2\bar{z}}{1+|z|^2 + \sqrt{-1}\tau}, 0 \right) df(\widehat{\Theta(t)})(\epsilon z, \epsilon^2 \tau) \frac{4^{n+1}dzd\tau}{(\tau^2 + (1+|z|^2)^2)^{n+1}} \\ &= \int_{B_{\epsilon^{-1}}(0)} (2\bar{z}_1(1+|z|^2 - \sqrt{-1}\tau), \dots, 2\bar{z}_n(1+|z|^2 - \sqrt{-1}\tau), 0) \\ &\quad \cdot \left(\sum_{i=1}^n \frac{\partial f(\widehat{\Theta(t)})}{\partial a_i} \epsilon a_i + \sum_{i=1}^n \frac{\partial f(\widehat{\Theta(t)})}{\partial b_i} \epsilon b_i + \frac{\partial f(\widehat{\Theta(t)})}{\partial \tau} \epsilon^2 \tau \right) \frac{4^{n+1}dzd\tau}{(\tau^2 + (1+|z|^2)^2)^{n+2}} \\ &= 2\epsilon \int_{B_{\epsilon^{-1}}(0)} \left(\frac{\partial f(\widehat{\Theta(t)})}{\partial a_1} a_1^2 - \sqrt{-1} \frac{\partial f(\widehat{\Theta(t)})}{\partial b_1} b_1^2, \dots, \frac{\partial f(\widehat{\Theta(t)})}{\partial a_n} a_n^2 - \sqrt{-1} \frac{\partial f(\widehat{\Theta(t)})}{\partial b_n} b_n^2, 0 \right) \\ &\quad \cdot \frac{4^{n+1}(1+|z|^2)}{(\tau^2 + (1+|z|^2)^2)^{n+2}} dzd\tau \\ &= \epsilon A_1 \left(\frac{\partial f(\widehat{\Theta(t)})}{\partial a_1} - \sqrt{-1} \frac{\partial f(\widehat{\Theta(t)})}{\partial b_1}, \dots, \frac{\partial f(\widehat{\Theta(t)})}{\partial a_n} - \sqrt{-1} \frac{\partial f(\widehat{\Theta(t)})}{\partial b_n}, 0 \right) + O(\epsilon^{2n+1}), \end{aligned}$$

where A_1 is given in (4.38). By (4.32) and symmetry, we have

$$(4.40) \quad \begin{aligned} II_1 - \langle II_1, \widehat{\Theta(t)} \widehat{\Theta(t)} \rangle &= \frac{1}{2} \int_{B_{\epsilon^{-1}}(0)} \left(\frac{2z}{1+|z|^2 - \sqrt{-1}\tau}, 0 \right) (\nabla df)(\widehat{\Theta(t)})(\epsilon z, \epsilon^2 \tau), (\epsilon z, \epsilon^2 \tau) \frac{4^{n+1}dzd\tau}{(\tau^2 + (1+|z|^2)^2)^{n+1}} \\ &= \int_{B_{\epsilon^{-1}}(0)} (2z_1(1+|z|^2 + \sqrt{-1}\tau), \dots, 2z_n(1+|z|^2 + \sqrt{-1}\tau), 0) \\ &\quad \cdot \frac{1}{2} \left(\sum_{i,j=1}^n \frac{\partial^2 f(\widehat{\Theta(t)})}{\partial a_i \partial a_j} \epsilon^2 a_i a_j + \sum_{i,j=1}^n \frac{\partial^2 f(\widehat{\Theta(t)})}{\partial b_i \partial b_j} \epsilon^2 b_i b_j + \sum_{i=1}^n \frac{\partial^2 f(\widehat{\Theta(t)})}{\partial a_i \partial \tau} \epsilon^3 a_i \tau \right. \\ &\quad \left. + \sum_{i=1}^n \frac{\partial^2 f(\widehat{\Theta(t)})}{\partial b_i \partial \tau} \epsilon^3 b_i \tau + \frac{\partial^2 f(\widehat{\Theta(t)})}{\partial \tau^2} \epsilon^4 \tau^2 \right) \frac{4^{n+1}dzd\tau}{(\tau^2 + (1+|z|^2)^2)^{n+1}} \\ &= 0. \end{aligned}$$

Similarly, we have

$$(4.41) \quad II_2 - \langle II_2, \widehat{\Theta(t)} \widehat{\Theta(t)} \rangle = 0.$$

Using (4.29)-(4.41), we can conclude that

$$\begin{aligned}
 (4.42) \quad & b - \langle b, (\widehat{\Theta(t)}, \widehat{\Theta(t)}) \rangle (\widehat{\Theta(t)}, \widehat{\Theta(t)}) \\
 &= \epsilon A_1 \left(\frac{\partial f(\widehat{\Theta(t)})}{\partial a_1} + \sqrt{-1} \frac{\partial f(\widehat{\Theta(t)})}{\partial b_1}, \dots, \frac{\partial f(\widehat{\Theta(t)})}{\partial a_n} + \sqrt{-1} \frac{\partial f(\widehat{\Theta(t)})}{\partial b_n}, 0, \right. \\
 & \quad \left. \frac{\partial f(\widehat{\Theta(t)})}{\partial a_1} - \sqrt{-1} \frac{\partial f(\widehat{\Theta(t)})}{\partial b_1}, \dots, \frac{\partial f(\widehat{\Theta(t)})}{\partial a_n} - \sqrt{-1} \frac{\partial f(\widehat{\Theta(t)})}{\partial b_n}, 0 \right) + E_4
 \end{aligned}$$

with errors

$$\begin{aligned}
 (4.43) \quad |E_4| &\leq C\epsilon^3 + C\|v - 1\|_{S^2_1(S^{2n+1}, \theta_0)} (F_2(t)^{\frac{1}{2}} + \|f - f(\widehat{\Theta(t)})\|_{L^2(S^{2n+1}, \theta_0)}) \\
 &\leq C\epsilon^3 + CF_2(t) + C\|f - f(\widehat{\Theta(t)})\|_{L^2(S^{2n+1}, \theta_0)}^2 \\
 &\leq C|\nabla_{\theta_0} f(\widehat{\Theta(t)})|_{\theta_0}^2 \epsilon^2 + C|b|^2 + C\epsilon^3
 \end{aligned}$$

where the second inequality follows from Lemma 4.7, and the third inequality follows from (4.34) and Lemma 4.6.

Case (ii). By (4.40) and symmetry, we have

$$\begin{aligned}
 (4.44) \quad \langle I_1, \widehat{\Theta(t)} \rangle &= \int_{B_{\epsilon^{-1}(0)}} \frac{1 - |z|^2 + \sqrt{-1}\tau}{1 + |z|^2 - \sqrt{-1}\tau} df(\widehat{\Theta(t)})(\epsilon z, \epsilon^2 \tau) \frac{4^{n+1} dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} \\
 &= \int_{B_{\epsilon^{-1}(0)}} (1 - |z|^2 + \sqrt{-1}\tau)(1 + |z|^2 + \sqrt{-1}\tau) \\
 & \quad \cdot \left(\sum_{i=1}^n \frac{\partial f(\widehat{\Theta(t)})}{\partial a_i} \epsilon a_i + \sum_{i=1}^n \frac{\partial f(\widehat{\Theta(t)})}{\partial b_i} \epsilon b_i + \frac{\partial f(\widehat{\Theta(t)})}{\partial \tau} \epsilon^2 \tau \right) \frac{4^{n+1} dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+2}} \\
 &= 0.
 \end{aligned}$$

Similarly, we have

$$(4.45) \quad \langle I_2, \widehat{\Theta(t)} \rangle = 0.$$

On the other hand, by (4.40) and symmetry, we have

$$\begin{aligned}
(4.46) \quad \langle II_1, \widehat{\Theta}(t) \rangle &= \frac{1}{2} \int_{B_{\epsilon^{-1}}(0)} \frac{1 - |z|^2 + \sqrt{-1}\tau}{1 + |z|^2 - \sqrt{-1}\tau} (\nabla df)(\widehat{\Theta}(t))((\epsilon z, \epsilon^2 \tau), (\epsilon z, \epsilon^2 \tau)) \frac{4^{n+1} dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} \\
&= \frac{1}{2} \int_{B_{\epsilon^{-1}}(0)} (1 - |z|^2 + \sqrt{-1}\tau)(1 + |z|^2 + \sqrt{-1}\tau) \\
&\quad \cdot \frac{1}{2} \left(\sum_{i,j=1}^n \frac{\partial^2 f(\widehat{\Theta}(t))}{\partial a_i \partial a_j} \epsilon^2 a_i a_j + \sum_{i,j=1}^n \frac{\partial^2 f(\widehat{\Theta}(t))}{\partial b_i \partial b_j} \epsilon^2 b_i b_j + \sum_{i=1}^n \frac{\partial^2 f(\widehat{\Theta}(t))}{\partial a_i \partial \tau} \epsilon^3 a_i \tau \right. \\
&\quad \left. + \sum_{i=1}^n \frac{\partial^2 f(\widehat{\Theta}(t))}{\partial b_i \partial \tau} \epsilon^3 b_i \tau + \frac{\partial^2 f(\widehat{\Theta}(t))}{\partial \tau^2} \epsilon^4 \tau^2 \right) \frac{4^{n+1} dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+2}} \\
&= \frac{1}{2} \int_{B_{\epsilon^{-1}}(0)} (1 - |z|^4 - \tau^2) \cdot \frac{1}{2} \left(\sum_{i=1}^n \frac{\partial^2 f(\widehat{\Theta}(t))}{\partial a_i^2} \epsilon^2 a_i^2 \right. \\
&\quad \left. + \sum_{i=1}^n \frac{\partial^2 f(\widehat{\Theta}(t))}{\partial b_i^2} \epsilon^2 b_i^2 + \frac{\partial^2 f(\widehat{\Theta}(t))}{\partial \tau^2} \epsilon^4 \tau^2 \right) \frac{4^{n+1} dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+2}} \\
&= \frac{1}{2n} \epsilon^2 \Delta_{\theta_0} f(\widehat{\Theta}(t)) \int_{\mathbb{H}^n} \frac{4^n (1 - |z|^4 - \tau^2) |z|^2 dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+2}} \\
&\quad + C \epsilon^4 \int_{B_{\epsilon^{-1}}(0)} \frac{4^{n+1} (1 - |z|^4 - \tau^2) \tau^2 dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+2}} + O(\epsilon^{2n}) \\
&= -\epsilon^2 A_2 \Delta_{\theta_0} f(\widehat{\Theta}(t)) + O(\epsilon^4)
\end{aligned}$$

where

$$(4.47) \quad A_2 := \frac{1}{2n} \int_{\mathbb{H}^n} \frac{4^n (|z|^4 + \tau^2 - 1) |z|^2}{(\tau^2 + (1 + |z|^2)^2)^{n+2}} dz d\tau$$

since

$$\begin{aligned}
\left| \int_{\mathbb{H}^n \setminus B_{\epsilon^{-1}}(0)} \frac{(|z|^4 + \tau^2 - 1) |z|^2}{(\tau^2 + (1 + |z|^2)^2)^{n+2}} dz d\tau \right| &\leq \int_{\mathbb{H}^n \setminus B_{\epsilon^{-1}}(0)} \frac{|z|^2}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} dz d\tau \\
&\leq C \int_0^\infty \frac{d\tau}{1 + \tau^2} \int_{\epsilon^{-1}}^\infty \frac{r^{2n+1}}{(1 + r^2)^{2n}} dr = O(\epsilon^{2n-2}).
\end{aligned}$$

Note that A_2 is positive because

$$\begin{aligned}
\int_{\mathbb{H}^n} \frac{(|z|^4 + \tau^2 - 1) |z|^2}{(\tau^2 + (1 + |z|^2)^2)^{n+2}} dz d\tau &= C \int_{\{r^4 + \tau^2 \geq 0\}} \frac{(r^4 + \tau^2 - 1) r^{2n+1}}{(\tau^2 + (1 + r^2)^2)^{n+2}} dr d\tau \\
&= C \int_{\{r^2 + \tau^2 \geq 0, r \geq 0\}} \frac{(r^2 + \tau^2 - 1) r^n}{(r^2 + \tau^2 + 2r + 1)^{n+2}} dr d\tau \geq \frac{C}{2^{n+2}} \int_{\{r^2 + \tau^2 \geq 0, r \geq 0\}} \frac{(r^2 + \tau^2 - 1) r^n}{(r^2 + \tau^2 + 1)^{n+2}} dr d\tau \\
&= \frac{C}{2^{n+2}} \int_0^\pi \int_0^\infty \frac{(r^2 - 1)(r \sin \theta)^n}{(r^2 + 1)^{n+2}} r dr d\theta = \frac{C}{2^{n+2}} \int_0^\pi \sin^n \theta d\theta \int_0^\infty \frac{(r^2 - 1) r^{n+1}}{(r^2 + 1)^{n+2}} dr,
\end{aligned}$$

where we have used the change of variables $r^2 \mapsto r$ in the second equality, and we have changed the coordinates (r, τ) to the polar coordinates (r, θ) in the third

equality. To see that the last term is positive, we note that

$$\begin{aligned}
\int_0^\infty \frac{(r^2 - 1)r^{n+1}}{(r^2 + 1)^{n+2}} dr &= \int_1^\infty \frac{(r^2 - 1)r^{n+1}}{(r^2 + 1)^{n+2}} dr + \int_0^1 \frac{(r^2 - 1)r^{n+1}}{(r^2 + 1)^{n+2}} dr \\
&= \int_1^\infty \frac{(r^2 - 1)r^{n+1}}{(r^2 + 1)^{n+2}} dr + \int_\infty^1 \frac{(\frac{1}{t^2} - 1)\frac{1}{t^{n+1}}}{(\frac{1}{t^2} + 1)^{n+2}} \left(-\frac{1}{t^2} dt\right) \\
&= \int_1^\infty \frac{(r^2 - 1)r^{n+1}}{(r^2 + 1)^{n+2}} dr + \int_1^\infty \frac{(1 - t^2)t^{n-1}}{(t^2 + 1)^{n+2}} dt \\
&= \int_1^\infty \frac{(r^2 - 1)(r^{n+1} - r^{n-1})}{(r^2 + 1)^{n+2}} dr = \int_1^\infty \frac{(r^2 - 1)^2 r^{n-1}}{(r^2 + 1)^{n+2}} dr > 0.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
(4.48) \quad \langle II_2, \widehat{\Theta}(t) \rangle &= \frac{1}{2} \int_{B_{\epsilon^{-1}}(0)} \frac{1 - |z|^2 - \sqrt{-1}\tau}{1 + |z|^2 + \sqrt{-1}\tau} (\nabla df)(\widehat{\Theta}(t))((\epsilon z, \epsilon^2 \tau), (\epsilon z, \epsilon^2 \tau)) \frac{4^{n+1} dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} \\
&= \frac{1}{2} \int_{B_{\epsilon^{-1}}(0)} (1 - |z|^2 - \sqrt{-1}\tau)(1 + |z|^2 - \sqrt{-1}\tau) \\
&\quad \cdot \frac{1}{2} \left(\sum_{i,j=1}^n \frac{\partial^2 f(\widehat{\Theta}(t))}{\partial a_i \partial a_j} \epsilon^2 a_i a_j + \sum_{i,j=1}^n \frac{\partial^2 f(\widehat{\Theta}(t))}{\partial b_i \partial b_j} \epsilon^2 b_i b_j + \sum_{i=1}^n \frac{\partial^2 f(\widehat{\Theta}(t))}{\partial a_i \partial \tau} \epsilon^3 a_i \tau \right. \\
&\quad \left. + \sum_{i=1}^n \frac{\partial^2 f(\widehat{\Theta}(t))}{\partial b_i \partial \tau} \epsilon^3 b_i \tau + \frac{\partial^2 f(\widehat{\Theta}(t))}{\partial \tau^2} \epsilon^4 \tau^2 \right) \frac{4^{n+1} dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+3}} \\
&= \frac{1}{2} \int_{B_{\epsilon^{-1}}(0)} (1 - |z|^4 - \tau^2) \cdot \frac{1}{2} \left(\sum_{i=1}^n \frac{\partial^2 f(\widehat{\Theta}(t))}{\partial a_i^2} \epsilon^2 a_i^2 \right. \\
&\quad \left. + \sum_{i=1}^n \frac{\partial^2 f(\widehat{\Theta}(t))}{\partial b_i^2} \epsilon^2 b_i^2 + \frac{\partial^2 f(\widehat{\Theta}(t))}{\partial \tau^2} \epsilon^4 \tau^2 \right) \frac{4^{n+1} dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+3}} \\
&= \frac{1}{2n} \epsilon^2 \Delta_{\theta_0} f(\widehat{\Theta}(t)) \int_{\mathbb{H}^n} \frac{4^n (1 - |z|^4 - \tau^2) |z|^2 dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+3}} \\
&\quad + C \epsilon^4 \int_{B_{\epsilon^{-1}}(0)} \frac{4^{n+1} (1 - |z|^4 - \tau^2) \tau^2 dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+3}} + O(\epsilon^{2n+2}) \\
&= -\epsilon^2 A_2 \Delta_{\theta_0} f(\widehat{\Theta}(t)) + O(\epsilon^4)
\end{aligned}$$

where A_2 is given in (4.47). Using (4.28)-(4.36), (4.44)-(4.48), Lemma 4.6 and 4.7, we have

$$(4.49) \quad \langle b, (\widehat{\Theta}(t), \widehat{\Theta}(t)) \rangle = -2A_2 \alpha \Delta_{\theta_0} f(\widehat{\Theta}(t)) \epsilon^2 + E_5,$$

with error

$$(4.50) \quad |E_5| \leq C |\nabla_{\theta_0} f(\widehat{\Theta}(t))|_{\theta_0}^2 \epsilon^2 + C |b|^2 + C \epsilon^3.$$

Now we have the estimate

$$\begin{aligned}
(4.51) \quad |b|^2 &= |b - \langle b, (\widehat{\Theta}(t), \widehat{\Theta}(t)) \rangle (\widehat{\Theta}(t), \widehat{\Theta}(t))|^2 + C \langle b, (\widehat{\Theta}(t), \widehat{\Theta}(t)) \rangle^2 \\
&\leq C |\nabla_{\theta_0} f(\widehat{\Theta}(t))|_{\theta_0}^2 \epsilon^2 + O(\epsilon^4) (1 + |\Delta_{\theta_0} f(\widehat{\Theta}(t))|^2) + C |b|^4
\end{aligned}$$

in view of (4.42), (4.43), (4.49) and (4.50). By Lemma 4.1 and 4.6, $|b|^2 \rightarrow 0$ as $t \rightarrow \infty$. Hence, we can choose sufficiently large t such that $C|b|^2 \leq 1/2$ so that the last term on the right hand side of (4.51) can be absorbed to the right hand side, i.e.

$$(4.52) \quad |b|^2 \leq C|\nabla_{\theta_0} f(\widehat{\Theta}(t))|_{\theta_0}^2 \epsilon^2 + O(\epsilon^4)(1 + |\Delta_{\theta_0} f(\widehat{\Theta}(t))|^2).$$

Now the assertion follows from (4.42), (4.43), (4.49), (4.50) and (4.52). \square

From (4.34), (4.52), Lemma 4.6, and Lemma 4.7, we obtain

$$(4.53) \quad F_2(t) = C|\nabla_{\theta_0} f(\widehat{\Theta}(t))|_{\theta_0}^2 \epsilon^2 + O(\epsilon^4) \text{ and } \|v-1\|_{S_1^2(S^{2n+1}, \theta_0)}^2 \leq C|\nabla_{\theta_0} f(\widehat{\Theta}(t))|_{\theta_0}^2 \epsilon^2 + O(\epsilon^3).$$

Lemma 4.9. *With $O(1) \leq C$ as $t \rightarrow \infty$, there holds*

$$\begin{aligned} b = \frac{\epsilon \text{Vol}(S^{2n+1}, \theta_0)}{n+1} & \left(\frac{dz_1}{dt}, \dots, \frac{dz_n}{dt}, -\frac{1}{2} \frac{dr}{dt} + \sqrt{-1} \epsilon \text{Im} \left(\frac{dz(t)}{dt} \cdot \overline{z(t)} \right) + \sqrt{-1} \frac{\epsilon}{2} \frac{d\tau}{dt}, \right. \\ & \left. \frac{d\bar{z}_1}{dt}, \dots, \frac{d\bar{z}_n}{dt}, -\frac{1}{2} \frac{dr}{dt} - \sqrt{-1} \epsilon \text{Im} \left(\frac{dz(t)}{dt} \cdot \overline{z(t)} \right) - \sqrt{-1} \frac{\epsilon}{2} \frac{d\tau}{dt} \right) \\ & + O(|\nabla_{\theta_0} f(\widehat{\Theta}(t))|_{\theta_0}^2 \epsilon^2 + O(\epsilon^3)). \end{aligned}$$

Proof. Using (3.3), we have

$$(4.54) \quad (n+1)b = (n+1) \int_{S^{2n+1}} (x, \bar{x})(\alpha(t)f_\phi - R_h) dV_h = \int_{S^{2n+1}} (\xi, \bar{\xi}) dV_h = (X, \bar{X}) + I,$$

where X is the vector given in (3.16) and

$$I = \int_{S^{2n+1}} (\xi, \bar{\xi})(v^{2+\frac{2}{n}} - 1) dV_{\theta_0}$$

which can be estimated as follows:

$$(4.55) \quad \begin{aligned} |I| & \leq C\|\xi\|_{L^\infty} \|v^{2+\frac{2}{n}} - 1\|_{S_1^2(S^{2n+1}, \theta_0)} \leq CF_2^{\frac{1}{2}} \|v-1\|_{S_1^2(S^{2n+1}, \theta_0)} \\ & \leq C(F_2 + \|v-1\|_{S_1^2(S^{2n+1}, \theta_0)}^2) \leq C|\nabla_{\theta_0} f(\widehat{\Theta}(t))|_{\theta_0}^2 \epsilon^2 + O(\epsilon^3) \end{aligned}$$

in view of (4.53), Lemma 2.3, and Lemma 3.1. Now the assertion follows from (3.17), (3.18), (4.54), and (4.55). \square

Lemma 4.10. *With $o(1) \rightarrow 0$ as $t \rightarrow \infty$, there holds*

$$\text{Vol}(S^{2n+1}, \theta_0)^2 - |\Theta(t)|^2 = (4\text{Vol}(S^{2n+1}, \theta_0)A_3 + o(1))\epsilon^2,$$

where A_3 is the positive number defined as in (4.59).

Proof. As before, we choose coordinates of \mathbb{H}^n such that $(0, \dots, 0, 1) = \widehat{\Theta}(t)$ corresponding to the point $\widehat{\Theta}(t)$. Hence $\phi(t)$ will have the usual representation. In particular,

$$(4.56) \quad \phi_{n+1} = \frac{1 - \epsilon^2|z|^2 + \sqrt{-1}\epsilon^2\tau}{1 + \epsilon^2|z|^2 - \sqrt{-1}\epsilon^2\tau} = 1 - 2\epsilon^2 \frac{|z|^2 - \sqrt{-1}\tau}{1 + \epsilon^2|z|^2 - \sqrt{-1}\epsilon^2\tau}.$$

Thus by symmetry we have

$$\begin{aligned}
(4.57) \quad \Theta(t)_{n+1} &= \int_{S^{2n+1}} \phi_{n+1}(t) dV_{\theta_0} \\
&= \text{Vol}(S^{2n+1}, \theta_0) - 2\epsilon^2 \int_{\mathbb{H}^n} \frac{|z|^2 - \sqrt{-1}\tau}{1 + \epsilon^2|z|^2 - \sqrt{-1}\epsilon^2\tau} \frac{dzd\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} \\
&= \text{Vol}(S^{2n+1}, \theta_0) - 2\epsilon^2 \int_{\mathbb{H}^n} \frac{\epsilon^2(|z|^4 + \tau^2) + |z|^2}{(1 + \epsilon^2|z|^2)^2 + \epsilon^4\tau^2} \frac{dzd\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}}.
\end{aligned}$$

Observe that

$$\begin{aligned}
(4.58) \quad & \int_{\mathbb{H}^n} \frac{\epsilon^2(|z|^4 + \tau^2) + |z|^2}{(1 + \epsilon^2|z|^2)^2 + \epsilon^4\tau^2} \frac{dzd\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} \\
&= \int_{\mathbb{H}^n} \frac{|z|^2 dzd\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} \\
&\quad - \int_{\mathbb{H}^n} \frac{\epsilon^2(|z|^4 - \tau^2) + \epsilon^4(|z|^6 + |z|^2\tau^2)}{(1 + \epsilon^2|z|^2)^2 + \epsilon^4\tau^2} \frac{dzd\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}}.
\end{aligned}$$

We are going to estimate the terms on the right hand side of (4.58). First note that

$$\begin{aligned}
(4.59) \quad A_3 &:= \int_{\mathbb{H}^n} \frac{|z|^2 dzd\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} \leq \int_{\{0 \leq |z| < \infty\}} \left(\int_{-\infty}^{\infty} \frac{d\tau}{1 + \tau^2} \right) \frac{|z|^2 dz}{(1 + |z|^2)^{2n}} \\
&= \pi \int_{\{0 \leq |z| < \infty\}} \frac{|z|^2 dz}{(1 + |z|^2)^{2n}} = C \int_0^\infty \frac{r^{2n+1} dr}{(1 + r^2)^{2n}} \\
&\leq C \int_0^1 \frac{r^{2n+1} dr}{(1 + r^2)^{2n}} + C \int_1^\infty \frac{dr}{r^{2n-1}} < \infty
\end{aligned}$$

when $n \geq 2$. On the other hand, we have

$$\begin{aligned}
(4.60) \quad & \int_{\mathbb{H}^n} \frac{\epsilon^2(|z|^4 - \tau^2) + \epsilon^4(|z|^6 + |z|^2\tau^2)}{(1 + \epsilon^2|z|^2)^2 + \epsilon^4\tau^2} \frac{dzd\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} \\
&= C \int_0^\infty \int_0^\infty \frac{\epsilon^2(r^4 - \tau^2) + \epsilon^4(r^6 + r^2\tau^2)}{(1 + \epsilon^2r^2)^2 + \epsilon^4\tau^2} \frac{r^{2n-1} dr d\tau}{(\tau^2 + (1 + r^2)^2)^{n+1}} \\
&= C \int_0^\infty \int_0^\infty \frac{\epsilon^2r^4 + \epsilon^4r^6}{(1 + \epsilon^2r^2)^2 + \epsilon^4\tau^2} \frac{r^{2n-1} dr d\tau}{(\tau^2 + (1 + r^2)^2)^{n+1}} \\
&\quad - C \int_0^\infty \int_0^\infty \frac{\epsilon^2\tau^2 - \epsilon^4r^2\tau^2}{(1 + \epsilon^2r^2)^2 + \epsilon^4\tau^2} \frac{r^{2n-1} dr d\tau}{(\tau^2 + (1 + r^2)^2)^{n+1}}
\end{aligned}$$

for some constant C depending only on n . Note that

$$\begin{aligned}
(4.61) \quad & \int_0^\infty \int_0^\infty \frac{\epsilon^2 r^4}{(1 + \epsilon^2 |z|^2)^2 + \epsilon^4 \tau^2} \frac{r^{2n-1} dr d\tau}{(\tau^2 + (1 + r^2)^2)^{n+1}} \\
& \leq \int_0^\infty \left(\int_0^1 + \int_1^{\frac{1}{\epsilon^2}} + \int_{\frac{1}{\epsilon^2}}^\infty \right) \frac{\epsilon^2 r^{2n+3}}{(1 + \epsilon^4 \tau^2)(\tau^2 + (1 + r^2)^2)^{n+1}} d\tau dr \\
& \leq \int_0^\infty \left(\int_0^1 \frac{1}{(1 + r^2)^{2n+2}} d\tau + \int_1^{\frac{1}{\epsilon^2}} \frac{1}{2\tau(1 + r^2)^{2n+1}} d\tau \right. \\
& \quad \left. + \int_{\frac{1}{\epsilon^2}}^\infty \frac{1}{2\epsilon^4 \tau^3(1 + r^2)^{2n+1}} d\tau \right) \epsilon^2 r^{2n+3} dr \\
& = \int_0^\infty \left(\frac{1}{(1 + r^2)^{2n+2}} + \frac{-\log \epsilon}{(1 + r^2)^{2n+1}} + \frac{1}{4(1 + r^2)^{2n+1}} \right) \epsilon^2 r^{2n+3} dr \\
& = \epsilon^2 \int_0^\infty \frac{r^{2n+3} dr}{(1 + r^2)^{2n+2}} - \epsilon^2 \log \epsilon \int_0^\infty \frac{r^{2n+3} dr}{(1 + r^2)^{2n+1}} + \frac{\epsilon^2}{4} \int_0^\infty \frac{r^{2n+3} dr}{(1 + r^2)^{2n+1}} \leq C\epsilon,
\end{aligned}$$

where we have used $\tau^2 + (1 + r^2)^2 \geq 2\tau(1 + r^2)$ in the second inequality, and the last inequality follows from $-\epsilon \log \epsilon \leq C$ for ϵ being sufficiently small and

$$(4.62) \quad \int_0^\infty \frac{r^k dr}{(1 + r^2)^l} \leq \int_0^1 r^k dr + \int_1^\infty \frac{dr}{r^{2l-k}} = \frac{1}{k+1} + \frac{1}{2l-k-1} \text{ if } 2l-k \geq 2 \text{ and } k+1 > 0.$$

Similarly, we can estimate

$$\begin{aligned}
(4.63) \quad & \int_0^\infty \int_0^\infty \frac{\epsilon^4 r^6}{(1 + \epsilon^2 r^2)^2 + \epsilon^4 \tau^2} \frac{r^{2n-1} dr d\tau}{(\tau^2 + (1 + r^2)^2)^{n+1}} \\
& \leq \int_0^\infty \left(\int_0^{\frac{1}{\epsilon^2}} + \int_{\frac{1}{\epsilon^2}}^\infty \right) \frac{\epsilon^4 r^{2n+5}}{(1 + \epsilon^4 \tau^2)(\tau^2 + (1 + r^2)^2)^{n+1}} d\tau dr \\
& \leq \int_0^\infty \left(\int_0^{\frac{1}{\epsilon^2}} d\tau + \int_{\frac{1}{\epsilon^2}}^\infty \frac{1}{\epsilon^4 \tau^2} d\tau \right) \frac{\epsilon^4 r^{2n+5}}{(1 + r^2)^{2n+2}} dr \\
& = 2\epsilon^2 \int_0^\infty \frac{r^{2n+5} dr}{(1 + r^2)^{2n+2}} \leq C\epsilon,
\end{aligned}$$

where the last inequality follows from (4.62). Note also that

$$\begin{aligned}
(4.64) \quad & \int_0^\infty \int_0^\infty \frac{\epsilon^2 \tau^2}{(1 + \epsilon^2 r^2)^2 + \epsilon^4 \tau^2} \frac{r^{2n-1} dr d\tau}{(\tau^2 + (1 + r^2)^2)^{n+1}} \\
& \leq \int_0^\infty \left(\int_0^{\frac{1}{\epsilon^{3/2}}} + \int_{\frac{1}{\epsilon^{3/2}}}^\infty \right) \frac{\epsilon^2 r^{2n-1}}{(1 + \epsilon^4 \tau^2)(\tau^2 + (1 + r^2)^2)^n} d\tau dr \\
& \leq \int_0^\infty \left(\int_0^{\frac{1}{\epsilon^{3/2}}} \frac{1}{(1 + r^2)^{2n}} d\tau + \int_{\frac{1}{\epsilon^{3/2}}}^\infty \frac{1}{\epsilon^4 \tau^3(1 + r^2)^{2n-1}} d\tau \right) \epsilon^2 r^{2n-1} dr \\
& = \epsilon^{1/2} \int_0^\infty \frac{r^{2n-1} dr}{(1 + r^2)^{2n}} + \frac{\epsilon}{2} \int_0^\infty \frac{r^{2n-1} dr}{(1 + r^2)^{2n-1}} \leq C\epsilon^{1/2}
\end{aligned}$$

where we have used $\tau^2 + (1 + r^2)^2 \geq 2\tau(1 + r^2)$ in the second inequality, and the last inequality follows from (4.62). On the other hand, we can estimate

$$\begin{aligned}
 & \int_0^\infty \int_0^\infty \frac{\epsilon^4 r^2 \tau^2}{(1 + \epsilon^2 r^2)^2 + \epsilon^4 \tau^2} \frac{r^{2n-1} dr d\tau}{(\tau^2 + (1 + r^2)^2)^{n+1}} \\
 & \leq \int_0^\infty \left(\int_0^{\frac{1}{\epsilon^2}} + \int_{\frac{1}{\epsilon^2}}^\infty \right) \frac{\epsilon^4 r^{2n+1}}{(1 + \epsilon^4 \tau^2)(1 + r^2)^{2n}} d\tau dr \\
 (4.65) \quad & \leq \int_0^\infty \left(\int_0^{\frac{1}{\epsilon^2}} d\tau + \int_{\frac{1}{\epsilon^2}}^\infty \frac{1}{\epsilon^4 \tau^2} d\tau \right) \frac{\epsilon^4 r^{2n+1}}{(1 + r^2)^{2n}} dr \\
 & = 2\epsilon^2 \int_0^\infty \frac{r^{2n+1} dr}{(1 + r^2)^{2n}} \leq C\epsilon,
 \end{aligned}$$

where the last inequality follows from (4.62). Combining (4.57)-(4.65), we conclude that

$$(4.66) \quad \Theta(t)_{n+1} = \text{Vol}(S^{2n+1}, \theta_0) - 2A_3\epsilon^2 + o(\epsilon^2).$$

From this, we have

$$\begin{aligned}
 \text{Vol}(S^{2n+1}, \theta_0)^2 - |\Theta(t)|^2 &= (\text{Vol}(S^{2n+1}, \theta_0) + \Theta(t)_{n+1})(\text{Vol}(S^{2n+1}, \theta_0) - \Theta(t)_{n+1}) \\
 &= (2\text{Vol}(S^{2n+1}, \theta_0) + o(1))(2A_3\epsilon^2 + o(\epsilon^2)) \\
 &= (4\text{Vol}(S^{2n+1}, \theta_0)A_3 + o(1))\epsilon^2,
 \end{aligned}$$

as required. \square

Lemma 4.11. *With $o(1) \rightarrow 0$ as $t \rightarrow \infty$, there hold*

$$\begin{aligned}
 \frac{d\Theta(t)_i}{dt} &= (A_4 + o(1))\epsilon^2 \frac{dz_i(t)}{dt} + o(1)\epsilon^2 z_i(t) \frac{dr(t)}{dt}, \quad 1 \leq i \leq n; \\
 \frac{d}{dt}(\text{Vol}(S^{2n+1}, \theta_0)^2 - |\Theta(t)|^2) &= (\text{Vol}(S^{2n+1}, \theta_0)^2 - |\Theta(t)|^2) \left[\left(\frac{A_5}{2A_3} + o(1) \right) \epsilon \frac{dr(t)}{dt} + o(1)\epsilon \frac{d\tau(t)}{dt} \right],
 \end{aligned}$$

where A_3 is the positive constant defined as in (4.59), A_4 and A_5 are the positive constants defined as in (4.76) and (4.78) respectively.

Proof. As we have remarked before, we have

$$\phi(t) = \Psi \circ \delta_{q(t), r(t)} \circ \pi.$$

Differentiating the identity

$$\Theta(t) = (\Theta(t)_1, \dots, \Theta(t)_{n+1}) = \int_{S^{2n+1}} \phi(t) dV_{\theta_0} = \int_{\mathbb{H}^n} \Psi \circ \delta_{q(t), r(t)}(z, \tau) \frac{4^{n+1} dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}}$$

at time t , we obtain

$$(4.67) \quad \frac{d\Theta(t)}{dt} = \int_{\mathbb{H}^n} d\Psi_y \left(\frac{d}{dt} \delta_{q(t), r(t)}(z, \tau) \right) \frac{4^{n+1} dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}}$$

where $y = \delta_{q(t), r(t)}(z, \tau)$. By (3.11), we have

$$\begin{aligned}
 (4.68) \quad & \frac{d}{dt} \delta_{q(t), r(t)}(z, \tau) \\
 &= \frac{dr(t)}{dt} \sum_{j=1}^n \left(a_j \frac{\partial}{\partial a_j} + b_j \frac{\partial}{\partial b_j} \right) + 2 \frac{dr(t)}{dt} \left[r(t) \tau + \sum_{j=1}^n (a_j b_j(t) - b_j a_j(t)) \right] \frac{\partial}{\partial \tau} \\
 &+ \sum_{j=1}^n \left(\frac{da_j(t)}{dt} \frac{\partial}{\partial a_j} + \frac{db_j(t)}{dt} \frac{\partial}{\partial b_j} \right) + \left[2r(t) \sum_{j=1}^n \left(a_j \frac{db_j(t)}{dt} - b_j \frac{da_j(t)}{dt} \right) + \frac{d\tau(t)}{dt} \right] \frac{\partial}{\partial \tau}.
 \end{aligned}$$

Let $\epsilon = 1/r(t)$. Combining (3.5), (3.6), (4.67) and (4.68), we obtain by symmetry that

$$\begin{aligned}
 (4.69) \quad & \frac{d\Theta(t)_{n+1}}{dt} = \int_{\mathbb{H}^n} \left\{ \frac{dr(t)}{dt} \sum_{j=1}^n \left[a_j \left(\frac{-4\epsilon^3 a_j}{(\epsilon^2 + |z|^2 - \sqrt{-1}\tau)^2} \right) + b_j \left(\frac{-4\epsilon^3 b_j}{(\epsilon^2 + |z|^2 - \sqrt{-1}\tau)^2} \right) \right] \right. \\
 &+ 2 \frac{dr(t)}{dt} \left[r(t) \tau + \sum_{j=1}^n (a_j b_j(t) - b_j a_j(t)) \right] \frac{2\sqrt{-1}\epsilon^4}{(\epsilon^2 + |z|^2 - \sqrt{-1}\tau)^2} \\
 &+ \sum_{j=1}^n \left[\frac{da_j(t)}{dt} \left(\frac{-4\epsilon^3 a_j}{(\epsilon^2 + |z|^2 - \sqrt{-1}\tau)^2} \right) + \frac{db_j(t)}{dt} \left(\frac{-4\epsilon^3 b_j}{(\epsilon^2 + |z|^2 - \sqrt{-1}\tau)^2} \right) \right] \\
 &\left. + \left[2r(t) \sum_{j=1}^n \left(a_j \frac{db_j(t)}{dt} - b_j \frac{da_j(t)}{dt} \right) + \frac{d\tau(t)}{dt} \right] \frac{2\sqrt{-1}\epsilon^4}{(\epsilon^2 + |z|^2 - \sqrt{-1}\tau)^2} \right\} \cdot \frac{4^{n+1} dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} \\
 &= -\epsilon^3 \frac{dr(t)}{dt} \int_{\mathbb{H}^n} \frac{4|z|^2}{(\epsilon^2 + |z|^2 - \sqrt{-1}\tau)^2} \cdot \frac{4^{n+1} dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} \\
 &+ \epsilon^4 \frac{d\tau(t)}{dt} \int_{\mathbb{H}^n} \frac{2\sqrt{-1}}{(\epsilon^2 + |z|^2 - \sqrt{-1}\tau)^2} \cdot \frac{4^{n+1} dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} \\
 &= -4\epsilon^3 \frac{dr(t)}{dt} \int_{\mathbb{H}^n} \frac{|z|^2 [(\epsilon^2 + |z|^2)^2 - \tau^2]}{[(\epsilon^2 + |z|^2)^2 + \tau^2]^2} \cdot \frac{4^{n+1} dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} \\
 &+ 2\epsilon^4 \sqrt{-1} \frac{d\tau(t)}{dt} \int_{\mathbb{H}^n} \frac{[(\epsilon^2 + |z|^2)^2 - \tau^2]}{[(\epsilon^2 + |z|^2)^2 + \tau^2]^2} \cdot \frac{4^{n+1} dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} \\
 &= -4\epsilon^3 \frac{dr(t)}{dt} \int_{\mathbb{H}^n} \left(\frac{|z|^2}{(\epsilon^2 + |z|^2)^2 + \tau^2} - \frac{2|z|^2 \tau^2}{[(\epsilon^2 + |z|^2)^2 + \tau^2]^2} \right) \cdot \frac{4^{n+1} dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} \\
 &+ 2\epsilon^4 \sqrt{-1} \frac{d\tau(t)}{dt} \int_{\mathbb{H}^n} \left(\frac{1}{(\epsilon^2 + |z|^2)^2 + \tau^2} - \frac{2\tau^2}{[(\epsilon^2 + |z|^2)^2 + \tau^2]^2} \right) \cdot \frac{4^{n+1} dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}}
 \end{aligned}$$

and, for $i = 1, \dots, n$,

(4.70)

$$\begin{aligned}
\frac{d\Theta(t)_i}{dt} &= \int_{\mathbb{H}^n} \left\{ \frac{dr(t)}{dt} \sum_{j=1}^n \left[a_j \left(\frac{2\epsilon^2 \delta_{ij}}{\epsilon^2 + |z|^2 - \sqrt{-1}\tau} - \frac{4\epsilon^2(a_i + \sqrt{-1}b_i)a_j}{(\epsilon^2 + |z|^2 - \sqrt{-1}\tau)^2} \right) \right. \right. \\
&\quad \left. \left. + b_j \left(\frac{2\epsilon^2 \sqrt{-1} \delta_{ij}}{\epsilon^2 + |z|^2 - \sqrt{-1}\tau} - \frac{4\epsilon^2(a_i + \sqrt{-1}b_i)b_j}{(\epsilon^2 + |z|^2 - \sqrt{-1}\tau)^2} \right) \right] \right. \\
&\quad + 2 \frac{dr(t)}{dt} \left[r(t)\tau + \sum_{j=1}^n (a_j b_j(t) - b_j a_j(t)) \right] \frac{2\sqrt{-1}\epsilon^3(a_i + \sqrt{-1}b_i)}{(\epsilon^2 + |z|^2 - \sqrt{-1}\tau)^2} \\
&\quad + \sum_{j=1}^n \left[\frac{da_j(t)}{dt} \left(\frac{2\epsilon^2 \delta_{ij}}{\epsilon^2 + |z|^2 - \sqrt{-1}\tau} - \frac{4\epsilon^2(a_i + \sqrt{-1}b_i)a_j}{(\epsilon^2 + |z|^2 - \sqrt{-1}\tau)^2} \right) \right. \\
&\quad \left. + \frac{db_j(t)}{dt} \left(\frac{2\epsilon^2 \sqrt{-1} \delta_{ij}}{\epsilon^2 + |z|^2 - \sqrt{-1}\tau} - \frac{4\epsilon^2(a_i + \sqrt{-1}b_i)b_j}{(\epsilon^2 + |z|^2 - \sqrt{-1}\tau)^2} \right) \right] \\
&\quad + \left[2r(t) \sum_{j=1}^n \left(a_j \frac{db_j(t)}{dt} - b_j \frac{da_j(t)}{dt} \right) + \frac{d\tau(t)}{dt} \right] \frac{2\sqrt{-1}\epsilon^3(a_i + \sqrt{-1}b_i)}{(\epsilon^2 + |z|^2 - \sqrt{-1}\tau)^2} \left. \right\} \frac{4^{n+1}dzd\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} \\
&= 4\sqrt{-1}\epsilon^3 \frac{dr(t)}{dt} \int_{\mathbb{H}^n} \frac{(a_i^2 b_i(t) - \sqrt{-1}b_i^2 a_i(t))}{(\epsilon^2 + |z|^2 - \sqrt{-1}\tau)^2} \cdot \frac{4^{n+1}dzd\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} \\
&\quad + 2\epsilon^2 \left(\frac{da_i(t)}{dt} + \sqrt{-1} \frac{db_i(t)}{dt} \right) \int_{\mathbb{H}^n} \frac{1}{(\epsilon^2 + |z|^2 - \sqrt{-1}\tau)} \cdot \frac{4^{n+1}dzd\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} \\
&\quad - 4\epsilon^2 \frac{da_i(t)}{dt} \int_{\mathbb{H}^n} \frac{a_i^2}{(\epsilon^2 + |z|^2 - \sqrt{-1}\tau)^2} \cdot \frac{4^{n+1}dzd\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} \\
&\quad - 4\epsilon^2 \sqrt{-1} \frac{db_i(t)}{dt} \int_{\mathbb{H}^n} \frac{b_i^2}{(\epsilon^2 + |z|^2 - \sqrt{-1}\tau)^2} \cdot \frac{4^{n+1}dzd\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} \\
&\quad + 4\sqrt{-1}\epsilon^3 r(t) \frac{db_i(t)}{dt} \int_{\mathbb{H}^n} \frac{a_i^2}{(\epsilon^2 + |z|^2 - \sqrt{-1}\tau)^2} \cdot \frac{4^{n+1}dzd\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} \\
&\quad + 4\epsilon^3 r(t) \frac{da_i(t)}{dt} \int_{\mathbb{H}^n} \frac{b_i^2}{(\epsilon^2 + |z|^2 - \sqrt{-1}\tau)^2} \cdot \frac{4^{n+1}dzd\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} \\
&= 2\epsilon^2 \frac{dz_i(t)}{dt} \int_{\mathbb{H}^n} \frac{1}{(\epsilon^2 + |z|^2 - \sqrt{-1}\tau)} \cdot \frac{4^{n+1}dzd\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} \\
&\quad + \frac{2\epsilon^3}{n} z_i(t) \frac{dr(t)}{dt} \int_{\mathbb{H}^n} \frac{|z|^2}{(\epsilon^2 + |z|^2 - \sqrt{-1}\tau)^2} \cdot \frac{4^{n+1}dzd\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} \\
&= 2\epsilon^2 \frac{dz_i(t)}{dt} \int_{\mathbb{H}^n} \frac{\epsilon^2 + |z|^2}{(\epsilon^2 + |z|^2)^2 + \tau^2} \cdot \frac{4^{n+1}dzd\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} \\
&\quad + \frac{2\epsilon^3}{n} z_i(t) \frac{dr(t)}{dt} \int_{\mathbb{H}^n} \frac{|z|^2[(\epsilon^2 + |z|^2)^2 - \tau^2]}{[(\epsilon^2 + |z|^2)^2 + \tau^2]^2} \cdot \frac{4^{n+1}dzd\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} \\
&= 2\epsilon^2 \frac{dz_i(t)}{dt} \int_{\mathbb{H}^n} \frac{\epsilon^2 + |z|^2}{(\epsilon^2 + |z|^2)^2 + \tau^2} \cdot \frac{4^{n+1}dzd\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} \\
&\quad + \frac{2\epsilon^3}{n} z_i(t) \frac{dr(t)}{dt} \int_{\mathbb{H}^n} \left(\frac{|z|^2}{(\epsilon^2 + |z|^2)^2 + \tau^2} - \frac{2|z|^2\tau^2}{[(\epsilon^2 + |z|^2)^2 + \tau^2]^2} \right) \cdot \frac{4^{n+1}dzd\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}}.
\end{aligned}$$

By Young's inequality

$$|z|^3|\tau|^{\frac{1}{2}} \leq \frac{3}{4}|z|^4 + \frac{1}{4}\tau^2 \leq |z|^4 + \tau^2,$$

we have

$$(4.71) \quad \left| \frac{1}{(\epsilon^2 + |z|^2)^2 + \tau^2} - \frac{1}{|z|^4 + \tau^2} \right| = \frac{2\epsilon^2|z|^2 + \epsilon^4}{[(\epsilon^2 + |z|^2)^2 + \tau^2](|z|^4 + \tau^2)} \leq \frac{2}{|z|^4 + \tau^2} \leq \frac{2}{|z|^3|\tau|^{\frac{1}{2}}}$$

and

$$(4.72) \quad \begin{aligned} \left| \frac{\tau^2}{[(\epsilon^2 + |z|^2)^2 + \tau^2]^2} - \frac{\tau^2}{(|z|^4 + \tau^2)^2} \right| &\leq \frac{\tau^2}{(\epsilon^2 + |z|^2)^2 + \tau^2} \left| \frac{1}{(\epsilon^2 + |z|^2)^2 + \tau^2} - \frac{1}{|z|^4 + \tau^2} \right| \\ &\quad + \frac{\tau^2}{|z|^4 + \tau^2} \left| \frac{1}{(\epsilon^2 + |z|^2)^2 + \tau^2} - \frac{1}{|z|^4 + \tau^2} \right| \\ &\leq 2 \left| \frac{|z|^2}{(\epsilon^2 + |z|^2)^2 + \tau^2} - \frac{|z|^2}{|z|^4 + \tau^2} \right| \leq \frac{2}{|z|^3|\tau|^{\frac{1}{2}}}. \end{aligned}$$

We also have

$$(4.73) \quad \begin{aligned} \left| \frac{|z|^2}{(\epsilon^2 + |z|^2)^2 + \tau^2} - \frac{|z|^2}{|z|^4 + \tau^2} \right| &= \frac{|z|^2(2\epsilon^2|z|^2 + \epsilon^4)}{[(\epsilon^2 + |z|^2)^2 + \tau^2](|z|^4 + \tau^2)} \\ &\leq \frac{2\epsilon^2(|z|^4 + 2\epsilon^2|z|^2)}{[(\epsilon^2 + |z|^2)^2 + \tau^2](|z|^4 + \tau^2)} \leq \frac{2\epsilon^2}{|z|^4 + \tau^2} \leq \frac{2\epsilon^2}{|z|^3|\tau|^{\frac{1}{2}}} \end{aligned}$$

which implies that

$$(4.74) \quad \begin{aligned} &\left| \frac{2|z|^2\tau^2}{[(\epsilon^2 + |z|^2)^2 + \tau^2]^2} - \frac{2|z|^2\tau^2}{(|z|^4 + \tau^2)^2} \right| \\ &\leq \frac{2\tau^2}{(\epsilon^2 + |z|^2)^2 + \tau^2} \left| \frac{|z|^2}{(\epsilon^2 + |z|^2)^2 + \tau^2} - \frac{|z|^2}{|z|^4 + \tau^2} \right| \\ &\quad + \frac{2\tau^2}{|z|^4 + \tau^2} \left| \frac{|z|^2}{(\epsilon^2 + |z|^2)^2 + \tau^2} - \frac{|z|^2}{|z|^4 + \tau^2} \right| \\ &\leq 4 \left| \frac{|z|^2}{(\epsilon^2 + |z|^2)^2 + \tau^2} - \frac{|z|^2}{|z|^4 + \tau^2} \right| \leq \frac{8\epsilon^2}{|z|^3|\tau|^{\frac{1}{2}}}. \end{aligned}$$

Since

$$(4.75) \quad \begin{aligned} &\int_{\mathbb{H}^n} \frac{1}{|z|^3|\tau|^{\frac{1}{2}}} \cdot \frac{dzd\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} \leq 2 \int_0^\infty \frac{d\tau}{\tau^{\frac{1}{2}}(1 + \tau^2)} \int_{\{|z| \geq 0\}} \frac{dz}{|z|^3(1 + |z|^2)^{2n-2}} \\ &\leq C \left(\int_0^1 \frac{d\tau}{\tau^{\frac{1}{2}}} + \int_1^\infty \frac{d\tau}{1 + \tau^2} \right) \left(\int_0^\infty \frac{r^{2n-4}dr}{(1 + r^2)^{2n}} \right) \leq C \end{aligned}$$

when $n \geq 2$ by (4.62), by the estimates (4.71)-(4.74), we can rewrite (4.70) as

$$\frac{d\Theta(t)_i}{dt} = (A_4\epsilon^2 + O(\epsilon^3)) \frac{dz_i(t)}{dt} + O(\epsilon^3)z_i(t) \frac{dr(t)}{dt}$$

where A_4 is the positive constant given by

$$(4.76) \quad A_4 = 2 \int_{\mathbb{H}^n} \frac{|z|^2}{|z|^4 + \tau^2} \cdot \frac{4^{n+1}dzd\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}},$$

and we can rewrite (4.69) as

$$(4.77) \quad \frac{d\Theta(t)_{n+1}}{dt} = (-\epsilon^3 A_5 + O(\epsilon^4)) \frac{dr(t)}{dt} + O(\epsilon^4) \frac{d\tau(t)}{dt}$$

where A_5 is the constant given by

$$(4.78) \quad A_5 = \int_{\mathbb{H}^n} \frac{4|z|^2(|z|^4 - \tau^2)}{(|z|^4 + \tau^2)^2} \cdot \frac{4^{n+1} dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}}.$$

Note that A_5 is positive. To see this, note that the right hand side of (4.78) can be written as

$$(4.79) \quad C \int_0^\infty \int_0^\infty \frac{(r^4 - \tau^2) r^{2n+1}}{(r^4 + \tau^2)[\tau^2 + (1 + r^2)^2]^{n+1}} dr d\tau$$

for some positive constant C . So it suffices to prove that the integral in (4.79) is positive. Let $u = r^2$ and $\tau = v$ and then using the polar coordinates $u = r \cos \theta$ and $v = r \sin \theta$, the integral can be written as

$$\begin{aligned} & \int_0^\infty \int_0^\infty \frac{(r^4 - \tau^2) r^{2n+1}}{(r^4 + \tau^2)[\tau^2 + (1 + r^2)^2]^{n+1}} dr d\tau \\ &= \frac{1}{2} \int_0^\infty \int_0^\infty \frac{(u^2 - v^2) u^n}{(u^2 + v^2)[v^2 + (1 + u)^2]^{n+1}} du dv \\ &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \int_0^\infty \frac{r^{n+1}(\cos^2 \theta - \sin^2 \theta) \cos^n \theta}{(r^2 + 2r \cos \theta + 1)^{n+1}} dr d\theta \\ &= \frac{1}{2} \left(\int_0^{\frac{\pi}{4}} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \right) \int_0^\infty \frac{r^{n+1}(\cos^2 \theta - \sin^2 \theta) \cos^n \theta}{(r^2 + 2r \cos \theta + 1)^{n+1}} dr d\theta \\ &= \frac{1}{2} \int_0^{\frac{\pi}{4}} \int_0^\infty \frac{r^{n+1}(\cos^2 \theta - \sin^2 \theta) \cos^n \theta}{(r^2 + 2r \cos \theta + 1)^{n+1}} dr d\theta \\ &\quad - \frac{1}{2} \int_{\frac{\pi}{4}}^0 \int_0^\infty \frac{r^{n+1}(\cos^2(\frac{\pi}{2} - \phi) - \sin^2(\frac{\pi}{2} - \phi)) \cos^n(\frac{\pi}{2} - \phi)}{(r^2 + 2r \cos(\frac{\pi}{2} - \phi) + 1)^{n+1}} dr d\phi \\ &= \frac{1}{2} \int_0^{\frac{\pi}{4}} \int_0^\infty r^{n+1}(\cos^2 \theta - \sin^2 \theta) \left[\frac{\cos^n \theta}{(r^2 + 2r \cos \theta + 1)^{n+1}} - \frac{\sin^n \theta}{(r^2 + 2r \sin \theta + 1)^{n+1}} \right] dr d\theta \\ &= \frac{1}{2} \int_0^{\frac{\pi}{4}} \int_0^\infty r^{n+1}(\cos^2 \theta - \sin^2 \theta) [h(r, \cos \theta) - h(r, \sin \theta)] dr d\theta, \end{aligned}$$

where $h(r, z) = \frac{z^n}{(r^2 + 2rz + 1)^{n+1}}$. Note that for $n \geq 2$ and $r > 0$, $h(r, z)$ is an increasing function in $z \in [0, 1]$. In particular, we have $h(r, \cos \theta) \geq h(r, \sin \theta)$ for $\theta \in [0, \pi/4]$, which implies that the integral in (4.79) is positive, and hence A_5 is positive.

Now by (4.77) and Lemma 4.10, we obtain

$$(4.80) \quad \frac{d\Theta(t)_{n+1}}{dt} = -(\text{Vol}(S^{2n+1}, \theta_0)^2 - |\Theta(t)|^2) \left[\left(\frac{A_5}{4\text{Vol}(S^{2n+1}, \theta_0)A_3} + o(1) \right) \epsilon \frac{dr(t)}{dt} + o(1) \epsilon \frac{d\tau(t)}{dt} \right].$$

By symmetry, $\Theta(t)_i = 0$ for $1 \leq i \leq n$. Thus by (4.66) and (4.80) we conclude that

$$\begin{aligned} \frac{d}{dt}(\text{Vol}(S^{2n+1}, \theta_0)^2 - |\Theta(t)|^2) &= -\widehat{\Theta(t)}_{n+1} \frac{d\Theta(t)_{n+1}}{dt} - \Theta(t)_{n+1} \frac{d\widehat{\Theta(t)}_{n+1}}{dt} \\ &= 2(\text{Vol}(S^{2n+1}, \theta_0) + o(1))(\text{Vol}(S^{2n+1}, \theta_0)^2 - |\Theta(t)|^2) \\ &\quad \cdot \left[\left(\frac{A_5}{4\text{Vol}(S^{2n+1}, \theta_0)A_3} + o(1) \right) \epsilon \frac{dr(t)}{dt} + o(1) \epsilon \frac{d\tau(t)}{dt} \right] \\ &= (\text{Vol}(S^{2n+1}, \theta_0)^2 - |\Theta(t)|^2) \left[\left(\frac{A_5}{2A_3} + o(1) \right) \epsilon \frac{dr(t)}{dt} + o(1) \epsilon \frac{d\tau(t)}{dt} \right], \end{aligned}$$

as required. \square

Proposition 4.12. *As $t \rightarrow \infty$, the contact form $\theta(t)$ concentrate at the critical point Q of f satisfying $\Delta_{\theta_0} f(Q) \leq 0$.*

Proof. It follows from Lemma 4.8-4.11 that

$$\begin{aligned} \frac{d}{dt}(\text{Vol}(S^{2n+1}, \theta_0)^2 - |\Theta(t)|^2) \\ (4.81) \quad &= (2A_5 \text{Vol}(S^{2n+1}, \theta_0) + o(1)) \epsilon^3 \frac{dr(t)}{dt} + o(1) \epsilon^3 \frac{d\tau(t)}{dt} \\ &= -(2(n+1)A_5 + o(1)) \epsilon^2 (b_{n+1} + b_{2n+2}) + O(|\nabla_{\theta_0} f(\widehat{\Theta(t)})|_{\theta_0}^2) \epsilon^4 + O(\epsilon^5) \\ &= 4(n+1)A_2 A_5 \alpha \epsilon^4 (\Delta_{\theta_0} f(\widehat{\Theta(t)})) + O(1) |\nabla_{\theta_0} f(\widehat{\Theta(t)})|_{\theta_0}^2 + O(\epsilon). \end{aligned}$$

By (4.81) and Lemma 4.10, we find

$$\left| \frac{d}{dt}(\text{Vol}(S^{2n+1}, \theta_0)^2 - |\Theta(t)|^2) \right| \leq C(\text{Vol}(S^{2n+1}, \theta_0)^2 - |\Theta(t)|^2)^2,$$

which yields

$$\text{Vol}(S^{2n+1}, \theta_0)^2 - |\Theta(t)|^2 \geq \frac{C_0}{t},$$

for some constant $C_0 > 0$, while by Lemma 4.10, we have

$$(4.82) \quad \epsilon^2 \geq \frac{C_1}{t},$$

for $t \geq t_0$ with some sufficiently large $t_0 > 0$ and a uniform constant $C_1 > 0$. It follows from Lemmas 4.8-4.11 that

$$\begin{aligned} \frac{d}{dt} f(\Theta(t)) &= \frac{d}{dt} f(\widehat{\Theta(t)}) = \frac{1}{2|\Theta(t)|} \sum_{i=1}^n \frac{\partial f}{\partial z_i}(\widehat{\Theta(t)}) \frac{d\Theta(t)_i}{dt} + O(\epsilon^3) \\ &\geq C\epsilon^2 (|f'(\widehat{\Theta(t)})|^2 + o(1)) \end{aligned}$$

where f' denotes the gradient of f with respect to the standard Riemannian metric on S^{2n+1} . This implies by (4.82) that

$$\left| \frac{d}{dt} f(\Theta(t)) \right| \geq \frac{C_2}{t} (|f'(\widehat{\Theta(t)})|^2 + o(1))$$

where $C_2 > 0$ and the error $o(1) \rightarrow 0$ as $t \rightarrow \infty$. Since t^{-1} is divergent, the flow $(\Theta(t))_{t \geq 0}$ must accumulate at a critical point of f . To see the critical point with $\Delta_{\theta_0} f(Q) \leq 0$ are the only possible limit points of $\Theta(t)$, first we observe that if $\Delta_{\theta_0} f(Q) > 0$, then by (4.81), we have, for sufficiently large t , $\frac{d}{dt}(\text{Vol}(S^{2n+1}, \theta_0)^2 -$

$|\Theta(t)|^2 > 0$. Hence it will contradict the fact that $\text{Vol}(S^{2n+1}, \theta_0)^2 - |\Theta(t)|^2 \rightarrow 0$ as $t \rightarrow \infty$.

Therefore, the shadow flow $(\Theta(t))_{t \geq 0}$ converges to a unique point $Q \in S^{2n+1}$. \square

Lemma 4.13. *Under the assumptions of Theorem 2.2, let $u(0) = u_0 \in C_f^\infty$ be initial data of the flow (2.4). Then as $t \rightarrow \infty$, we have*

$$E_f(u(t)) \rightarrow R_{\theta_0} \text{Vol}(S^{2n+1}, \theta_0)^{\frac{1}{n+1}} f(Q)^{-\frac{n}{n+1}},$$

where $Q = \lim_{t \rightarrow \infty} \Theta(t)$ is the unique limit of the shadow flow $\Theta(t)$ associated with $u(t)$.

Proof. Note that

$$(4.83) \quad E(u(t)) = E(v(t)) \rightarrow R_{\theta_0} \text{Vol}(S^{2n+1}, \theta_0) \quad \text{as } t \rightarrow \infty$$

by Lemma 2.3. On the other hand, by Lemma 2.3, we have

$$\int_{S^{2n+1}} f dV_\theta = \int_{S^{2n+1}} f_\phi dV_\theta \rightarrow f(Q) \text{Vol}(S^{2n+1}, \theta_0) \quad \text{as } t \rightarrow \infty.$$

Combining these, the assertion follows. \square

5. EXISTENCE OF CONFORMAL CONTACT FORM

In this section, for $p \in S^{2n+1}$, $0 < \epsilon < 1$, as before, we denote by $\phi_{-p, \epsilon}$ the projection with $-p$ at infinity, that is, p becomes the north pole in the coordinates. Define a map

$$j : S^{2n+1} \times (0, \infty) \ni (p, \epsilon) \mapsto u_{p, \epsilon} = |\det(d\phi_{-p, \epsilon})|^{\frac{n}{2n+2}} \in C_*^\infty$$

where

$$C_*^\infty := \left\{ 0 < u \in C^\infty(S^{2n+1}) : \theta = u^{\frac{2}{n}} \theta_0 \text{ satisfies } \int_{S^{2n+1}} u^{2+\frac{2}{n}} dV_{\theta_0} = \int_{S^{2n+1}} dV_{\theta_0} \right\}.$$

Also let $\theta_{p, \epsilon} = \phi_{p, \epsilon}^*(\theta_0) = u_{p, \epsilon}^{\frac{2}{n}} \theta_0$ to get

$$dV_{\theta_{p, \epsilon}} = u_{p, \epsilon}^{2+\frac{2}{n}} dV_{\theta_0} \rightharpoonup \text{Vol}(S^{2n+1}, \theta_0) \delta_p$$

in the weak sense of measures as $\epsilon \rightarrow 0$. For $\gamma \in \mathbb{R}$, denote by

$$L_\gamma = \{u \in C_*^\infty : E_f(u) \leq \gamma\},$$

the sub-level set of E_f . For convenience, labeling all the critical points of f by p_1, \dots, p_N such that $f(p_i) \leq f(p_j)$ for $1 \leq i \leq j \leq N$, we set

$$\beta_i = R_{\theta_0} \text{Vol}(S^{2n+1}, \theta_0)^{\frac{1}{n+1}} f(p_i)^{-\frac{n}{n+1}} = \lim_{\epsilon \rightarrow 0} E_f(u_{p_i, \epsilon}), \quad 1 \leq i \leq N.$$

In view of Proposition 4.12, under our assumption of f , minimum points of f cannot be concentration points, namely, the energy level where the concentration occurs is strictly less than β_1 . Without loss of generality, we assume all critical levels $f(p_i)$, $1 \leq i \leq N$, are different, so that there exists a $\nu_0 > 0$ such that $\beta_i - 2\nu_0 > \beta_{i+1}$, in fact, we can take $\nu_0 = \frac{1}{3} \min_{1 \leq i \leq N-1} \{\beta_i - \beta_{i+1}\} > 0$. In the following, denote by $u(t, u_0)$ the flow (2.4) with initial data $u_0 \in C_*^\infty$, and again denote the shadow flow by

$$\Theta(t, u_0) = \int_{S^{2n+1}} \phi(t, u_0) dV_{S^{2n+1}} \quad \text{with } \widehat{\Theta(t, u_0)} = \frac{\Theta(t, u_0)}{\|\Theta(t, u_0)\|} \text{ if } \|\Theta(t, u_0)\| \neq 0.$$

Our main purpose of this section is to set up the following:

- Proposition 5.1.** (i) If $\beta_1 < \beta_0 \leq \beta$, where β has been chosen as in (2.14), then the set L_{β_0} is contractible.
(ii) For any $0 < \nu \leq \nu_0$ and each $1 \leq i \leq N$, the sets $L_{\beta_i - \nu}$ and $L_{\beta_{i+1} + \nu}$ are homotopic equivalent.
(iii) For each critical point p_i of f where $\Delta_{\theta_0} f(p_i) > 0$, the sets $L_{\beta_i + \nu_0}$ and $L_{\beta_i - \nu_0}$ are homotopic equivalent.
(iv) For each critical point p_i of f where $\Delta_{\theta_0} f(p_i) < 0$, the set $L_{\beta_i + \nu_0}$ is homotopic to the set $L_{\beta_i - \nu_0}$ with $(2n + 1 - \text{ind}(f, p_i))$ -cell attached.

By assuming Proposition 5.1, we can complete the proof of our main theorem.

Proof of Theorem 1.4. By contradiction, suppose that the flow does not converge and f cannot be realized as the Webster scalar curvature of a contact form conformal to the standard contact form θ_0 of S^{2n+1} . Then Proposition 5.1 shows that, L_{β_0} is contractible for some suitable β_0 chosen in part (i) of Proposition 5.1; in addition, the flow gives a homotopy equivalence of the set L_{β_0} with a set E_∞ whose homotopy type consists of a point $\{p_0\}$ with cells of dimension $(2n + 1 - \text{ind}(f, p_i))$ attached for each critical point p_i of f where $\Delta_{\theta_0} f(p_i) < 0$.

From [3], Theorem 4.3 on page 36, we conclude that the identity

$$(5.1) \quad \sum_{i=0}^{2n+1} t^i m_i = 1 + (1+t) \sum_{i=0}^{2n+1} t^i k_i$$

holds with $k_i \geq 0$ and m_i is given as in (1.2). Equating the coefficients of t in the polynomials on the left and right hand side, we obtain (1.3), which violates the hypothesis in Theorem 1.4 and thus leads to the desired contradiction. \square

Remark. By forming the alternating sum of the terms in (1.3), which corresponds to setting $t = -1$ in (5.1), we obtain

$$\sum_{f'(x)=0, \Delta_{\theta_0} f(x) < 0} (-1)^{\text{ind}(f, x)} = -1,$$

which contradicts (1.1). From this, we see that Theorem 1.4 implies Theorem 1.3.

The rest of this section is devoted to proving Proposition 5.1. By the long existence of the flow (2.4) which was proved in part I, we can assume that, for any fixed initial data u_0 and any finite $T > 0$, there exists $C(T) > 0$ such that $\|u\|_{L^\infty([0, T] \times C^{4n+4}(S^{2n+1}))} \leq C(T)$.

Lemma 5.2. *Given any $T > 0$, let $u_i(t) = u(t, u_i^0)$ be the solutions to our flow (2.4) with initial data $u_i^0 \in C_f^\infty$ such that $\|u_i\|_{L^\infty([0, T] \times C^{4n+4}(S^{2n+1}))} \leq C(T)$, $i = 1, 2$. Then there exists a constant $C > 0$ depending on T , n and $\|u_i\|_{L^\infty([0, T] \times C^{4n+4}(S^{2n+1}))}$, $i = 1, 2$, such that*

$$\sup_{0 \leq t \leq T} \|u_1(t) - u_2(t)\|_{S_{4n+4}^2(S^{2n+1}, \theta_0)} \leq C \|u_1^0 - u_2^0\|_{S_{4n+4}^2(S^{2n+1}, \theta_0)}.$$

Proof. By the long existence of the flow (2.4) which was proved in part I, we know that $u_i(t)$, $i = 1, 2$ are smooth in any given finite time interval $[0, T]$. Moreover, by Lemma 2.8 in [17], there exists constant $C_i = C_i(T) > 0$ such that

$$(5.2) \quad C_i^{-1} \leq \|u_i(t)\|_{L^\infty(S^{2n+1} \times [0, T])} \leq C_i \text{ for } i = 1, 2.$$

For simplicity, we let $\theta_i = u_i(t)^{\frac{2}{n}}\theta_0$, $R_i = R_{\theta_i}$, and by (2.3) and (2.5) the factor $\alpha_i(t)$ can be expressed as

$$\alpha(u_i) = \alpha_i(t) = \frac{\int_{S^{2n+1}} \left((2 + \frac{2}{n}) |\nabla_{\theta_0} u_i(t)|_{\theta_0}^2 + R_{\theta_0} u_i(t)^2 \right) dV_{\theta_0}}{\int_{S^{2n+1}} f u_i(t)^{2+\frac{2}{n}} dV_{\theta_0}}$$

for $i = 1, 2$. If set $w = u_2 - u_1$, we can estimate the term $\alpha(u_2) - \alpha(u_1)$ as follows:

(5.3)

$$\begin{aligned} \alpha(u_2) - \alpha(u_1) &= \int_0^1 \frac{\partial}{\partial s} \alpha(u_1 + sw) ds \\ &= \int_0^1 \left[\frac{2 \int_{S^{2n+1}} \left((1-s) R_1 u_1^{1+\frac{2}{n}} + s R_2 u_2^{1+\frac{2}{n}} \right) w dV_{\theta_0}}{\int_{S^{2n+1}} f(u_1 + sw)^{2+\frac{2}{n}} dV_{\theta_0}} \right. \\ &\quad \left. - \left(2 + \frac{2}{n} \right) \frac{E(u_1 + sw)}{(\int_{S^{2n+1}} f(u_1 + sw)^{2+\frac{2}{n}} dV_{\theta_0})^2} \int_{S^{2n+1}} f(u_1 + sw)^{1+\frac{2}{n}} w dV_{\theta_0} \right] ds \\ &\leq C(\|R_1\|_{L^2(S^{2n+1}, \theta_1)} + \|R_2\|_{L^2(S^{2n+1}, \theta_2)} + E(u_1) + E(u_2)) \|w\|_{L^2(S^{2n+1}, \theta_0)} \\ &\leq C \|w\|_{L^2(S^{2n+1}, \theta_0)}, \end{aligned}$$

where we have used (5.2), Lemma 2.11 in [17], and the fact that $E(u_i) \leq \gamma$ for $i = 1, 2$, and $E(u_1 + sw) = E((1-s)u_1 + su_2) \leq (1-s)E(u_1) + sE(u_2) \leq E(u_1) + E(u_2)$. From (2.4) and (2.5), that is,

$$\frac{\partial u_i}{\partial t} = \frac{n}{2} (\alpha(u_i) f - R_i) u_i \quad \text{for } i = 1, 2,$$

and

$$-(2 + \frac{2}{n}) \Delta_{\theta_0} u_i + R_{\theta_0} u_i = R_i u_i^{1+\frac{2}{n}} \quad \text{for } i = 1, 2,$$

a direct computation yields

$$\begin{aligned} \frac{\partial w}{\partial t} &= \frac{\partial u_2}{\partial t} - \frac{\partial u_1}{\partial t} \\ &= \frac{n}{2} (R_1 u_1 - R_2 u_2) + \frac{n}{2} [\alpha(u_2) f u_2 - \alpha(u_1) f u_1] \\ &= \frac{n}{2} (R_1 u_1 - R_2 u_2) + \frac{n}{2} [\alpha(u_1) f w + \alpha(u_2) f u_2 - \alpha(u_1) f u_2] \\ &= \frac{n}{2} \left[u_2^{-\frac{2}{n}} [(R_{\theta_0} u_2 - R_2 u_2^{1+\frac{2}{n}}) - (R_{\theta_0} u_1 - R_1 u_1^{1+\frac{2}{n}})] \right. \\ &\quad \left. + [\alpha(u_1) f - R_1 u_1^{1+\frac{2}{n}} \left(\frac{u_2^{-\frac{2}{n}} - u_1^{-\frac{2}{n}}}{u_2 - u_1} \right) - R_{\theta_0} u_2^{-\frac{2}{n}}] w \right] \\ &\quad + \frac{n}{2} (\alpha(u_2) - \alpha(u_1)) u_2 f \\ &= \frac{n}{2} \left[\left(2 + \frac{2}{n} \right) u_2^{-\frac{2}{n}} (\Delta_{\theta_0} u_2 - \Delta_{\theta_0} u_1) + d(x, t) w \right] + \frac{n}{2} (\alpha(u_2) - \alpha(u_1)) u_2 f \\ &= \frac{n}{2} \left[\left(2 + \frac{2}{n} \right) u_2^{-\frac{2}{n}} \Delta_{\theta_0} w + d(x, t) w \right] + \frac{n}{2} (\alpha(u_2) - \alpha(u_1)) u_2 f, \end{aligned} \tag{5.4}$$

where $d(x, t) = \alpha(u_1)f + b(x, t) - R_{\theta_0}u_2^{-\frac{2}{n}}$ and $b(x, t) = -R_1u_1^{1+\frac{2}{n}}\left(\frac{u_2^{-\frac{2}{n}} - u_1^{-\frac{2}{n}}}{u_2 - u_1}\right)$.

Thus, from (5.4), we have

$$(5.5) \quad \begin{aligned} \frac{d}{dt} \left(\int_{S^{2n+1}} w^2 dV_{\theta_0} \right) &= \int_{S^{2n+1}} 2w \frac{\partial w}{\partial t} dV_{\theta_0} \\ &= n \int_{S^{2n+1}} \left(\left(2 + \frac{2}{n}\right) u_2^{-\frac{2}{n}} w \Delta_{\theta_0} w + d(x, t) w^2 \right) dV_{\theta_0} + n(\alpha(u_2) - \alpha(u_1)) \int_{S^{2n+1}} u_2 f w dV_{\theta_0}. \end{aligned}$$

By (5.2), Hölder's inequality and Young's inequality, we have

$$(5.6) \quad \begin{aligned} &\int_{S^{2n+1}} u_2^{-\frac{2}{n}} w \Delta_{\theta_0} w dV_{\theta_0} \\ &= - \int_{S^{2n+1}} u_2^{-\frac{2}{n}} |\nabla_{\theta_0} w|_{\theta_0}^2 dV_{\theta_0} + \frac{2}{n} \int_{S^{2n+1}} u_2^{-(1+\frac{2}{n})} w \langle \nabla_{\theta_0} u_2, \nabla_{\theta_0} w \rangle_{\theta_0} dV_{\theta_0} \\ &\leq -\frac{1}{2} C_2^{-\frac{2}{n}} \int_{S^{2n+1}} |\nabla_{\theta_0} w|_{\theta_0}^2 dV_{\theta_0} + C \int_{S^{2n+1}} w^2 dV_{\theta_0}. \end{aligned}$$

By (5.2), (5.3) and Lemma 2.11 in [17], one has

$$(5.7) \quad \left| (\alpha(u_2) - \alpha(u_1)) \int_{S^{2n+1}} u_2 f w dV_{\theta_0} \right| + \left| \int_{S^{2n+1}} d(x, t) w^2 dV_{\theta_0} \right| \leq C \int_{S^{2n+1}} w^2 dV_{\theta_0}.$$

Combining (5.5), (5.6) and (5.7), we obtain

$$\frac{d}{dt} \left(\int_{S^{2n+1}} w^2 dV_{\theta_0} \right) + (n+1) C_2^{-\frac{2}{n}} \int_{S^{2n+1}} |\nabla_{\theta_0} w|_{\theta_0}^2 dV_{\theta_0} \leq C_0 \int_{S^{2n+1}} w^2 dV_{\theta_0}$$

for some constant C_0 . Therefore, for any $t \in [0, T]$, we can integrate the above differential inequality from 0 to t to obtain

$$(5.8) \quad \int_{S^{2n+1}} w^2(t) dV_{\theta_0} \leq e^{C_0 t} \int_{S^{2n+1}} w^2(0) dV_{\theta_0}.$$

Next, for any $p \in \mathbb{N}$ with $p \leq 2n+2$, by (5.4) one has

$$\begin{aligned} \frac{d}{dt} \int_{S^{2n+1}} |(-\Delta_{\theta_0})^p w|^2 dV_{\theta_0} &= 2 \int_{S^{2n+1}} \frac{\partial w}{\partial t} (-\Delta_{\theta_0})^{2p} w dV_{\theta_0} \\ &= n \int_{S^{2n+1}} \left[\left(2 + \frac{2}{n}\right) u_2^{-\frac{2}{n}} \Delta_{\theta_0} w + d(x, t) w \right] (-\Delta_{\theta_0})^{2p} w dV_{\theta_0} \\ &\quad + n \int_{S^{2n+1}} (\alpha(u_2) - \alpha(u_1)) u_2 f (-\Delta_{\theta_0})^{2p} w dV_{\theta_0}. \end{aligned}$$

By Interpolation, Hölder's and Young's inequalities, we obtain

$$\begin{aligned} &\int_{S^{2n+1}} (-\Delta_{\theta_0})^{2p} w (u_2^{-\frac{2}{n}} \Delta_{\theta_0} w) dV_{\theta_0} \\ &\leq -\frac{1}{2} C_2^{-\frac{2}{n}} \int_{S^{2n+1}} |\nabla_{\theta_0} (-\Delta_{\theta_0})^p w|_{\theta_0}^2 dV_{\theta_0} + C \int_{S^{2n+1}} |(-\Delta_{\theta_0})^p w|^2 dV_{\theta_0} + C \int_{S^{2n+1}} w^2 dV_{\theta_0} \end{aligned}$$

and also

$$\left| \int_{S^{2n+1}} d(x, t) w (-\Delta_{\theta_0})^{2p} w dV_{\theta_0} \right| \leq C \int_{S^{2n+1}} |(-\Delta_{\theta_0})^p w|^2 dV_{\theta_0} + C \int_{S^{2n+1}} w^2 dV_{\theta_0}.$$

By (5.3) and integration by parts, we get

$$\begin{aligned}
& \left| \int_{S^{2n+1}} (-\Delta_{\theta_0})^{2p} w(\alpha(u_2) - \alpha(u_1)) u_2 f dV_{\theta_0} \right| \\
&= \left| \int_{S^{2n+1}} w(\alpha(u_2) - \alpha(u_1)) (-\Delta_{\theta_0})^{2p} (u_2 f) dV_{\theta_0} \right| \\
&\leq C \|w\|_{L^2(S^{2n+1}, \theta_0)} \left| \int_{S^{2n+1}} w (-\Delta_{\theta_0})^{2p} (u_2 f) dV_{\theta_0} \right| \leq C \|w\|_{L^2(S^{2n+1}, \theta_0)}^2.
\end{aligned}$$

Combining the above estimates, we obtain

$$\begin{aligned}
\frac{d}{dt} \int_{S^{2n+1}} |(-\Delta_{\theta_0})^p w|^2 dV_{\theta_0} &\leq C \int_{S^{2n+1}} |(-\Delta_{\theta_0})^p w|^2 dV_{\theta_0} + C \int_{S^{2n+1}} w^2 dV_{\theta_0} \\
&\leq C \int_{S^{2n+1}} |(-\Delta_{\theta_0})^p w|^2 dV_{\theta_0} + C e^{C_0 t} \int_{S^{2n+1}} w^2(0) dV_{\theta_0}
\end{aligned}$$

by (5.8). Integrating it from 0 to t , where $t \in [0, T]$, we get

$$\begin{aligned}
(5.9) \quad & \int_{S^{2n+1}} |(-\Delta_{\theta_0})^p w(t)|^2 dV_{\theta_0} \\
& \leq e^{Ct} \int_{S^{2n+1}} |(-\Delta_{\theta_0})^p w(0)|^2 dV_{\theta_0} + e^{(C+C_0)t} \left(\frac{C}{C_0}\right) \int_{S^{2n+1}} w^2(0) dV_{\theta_0}.
\end{aligned}$$

Therefore, by choosing $p = 2n + 2$, we can combine (5.8) and (5.9) to yield

$$\sup_{0 \leq t \leq T} \|w(t)\|_{S_{4n+4}^2(S^{2n+1}, \theta_0)} \leq C e^{Ct} \|w(0)\|_{S_{4n+4}^2(S^{2n+1}, \theta_0)}$$

as required. \square

Proof of Proposition 5.1 (i). Let β_0 be chosen above, i.e. $\beta_1 < \beta_0 \leq \beta$. For $u_0 \in L_{\beta_0}$, let $u(t, u_0)$ be the solution of the flow determined by the initial data u_0 . By Proposition 2.1, the energy E_f is decreasing along the flow. In particular, we have

$$E_f(u(t, u_0)) \leq \beta_0.$$

Now for sufficiently small $\epsilon > 0$, we claim that there exists $T_1(u_0, \epsilon) > 0$ which depends continuously on u_0 in the $S_{4n+4}^2(S^{2n+1}, \theta_0)$ topology and if $t > T_1 = T_1(u_0, \epsilon)$, we have

$$(5.10) \quad \|v - 1\|_{C_P^1(S^{2n+1})} < \epsilon.$$

To prove this claim, first note that we can choose T_2 large so that if $t \geq T_2$, then

$$(5.11) \quad \|v - 1\|_{C_P^1(S^{2n+1})} < \frac{1}{2}.$$

This is possible since $\|v - 1\|_{C^1(S^{2n+1}, \theta_0)} \rightarrow 0$ as $t \rightarrow \infty$ by Lemma 2.3. Thus it follows from the expression for $\Delta_{\theta_0} v$ as in (4.23) that, for some constant C_1 which depends on n and T_3 , the upper bounds of F_{4n+4} and $\alpha(t)$, the maximum of f as well as the constant we have found in Lemma 4.7,

$$(5.12) \quad \int_{S^{2n+1}} |-\Delta_{\theta_0} v|^{2n+2} dV_{\theta_0} \leq C_1 (F_2^{\frac{1}{2}} + \|f_\phi - f(\widehat{\Theta(t)})\|_{L^2(S^{2n+1}, \theta_0)}).$$

Second it follows from (5.11) and Lemma 4.7 that for $t \geq T_2$

$$(5.13) \quad \begin{aligned} \int_{S^{2n+1}} |v-1|^{2n+2} dV_{\theta_0} &\leq \left(\frac{1}{2}\right)^{2n} \int_{S^{2n+1}} |v-1|^2 dV_{\theta_0} \\ &\leq C_2 (F_2^{\frac{1}{2}} + \|f_\phi - f(\widehat{\Theta}(t))\|_{L^2(S^{2n+1}, \theta_0)}) \end{aligned}$$

for some constant $C_2 > 0$.

Then by Folland-Stein embedding theorem, there exists a constant $C_0 > 0$ depending only on the dimension n such that

$$(5.14) \quad \|v-1\|_{C_P^1(S^{2n+1})} \leq C_0 \left[\int_{S^{2n+1}} |-\Delta_{\theta_0} v|^{2n+2} dV_{\theta_0} + \int_{S^{2n+1}} |v-1|^{2n+2} dV_{\theta_0} \right]^{\frac{1}{2n+2}}.$$

Now we choose $T_3 > T_2$ such that the quantity $|o(1)| < 1$ in the Lemma 4.2 for $t > T_3$.

Choose $B = (n+3)M^{\frac{n}{n+1}} \text{Vol}(S^{2n+1}, \theta_0)^{\frac{n}{n+1}}$ where $M = \max_{S^{2n+1}} f$ and consider

$$(5.15) \quad g(t) = F_2(t) + B \left[E_f(u)(t) - R_{\theta_0} \text{Vol}(S^{2n+1}, \theta_0)^{\frac{1}{n+1}} f(Q)^{-\frac{n}{n+1}} \right],$$

where Q is the unique concentration point of the flow or the shadow flow $\Theta(t)$. It follows from Proposition 2.1 and Lemma 4.2 that

$$\begin{aligned} \frac{dg(t)}{dt} &= \frac{d}{dt} F_2(t) - \frac{Bn \int_{S^{2n+1}} (\alpha f - R_\theta)^2 u^{2+\frac{2}{n}} dV_{\theta_0}}{\left(\int_{S^{2n+1}} f u^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{n}{n+1}}} \\ &\leq (n+1+o(1))(nF_2(t) - 2G_2(t)) + o(1)F_2(t) - n(n+3)F_2(t) < 0 \end{aligned}$$

and $g(t) > 0$ for all $t \geq T_3$.

Now for any $\epsilon > 0$, since $\lim_{t \rightarrow \infty} \|f_\phi - f(\widehat{\Theta}(t))\|_{L^2(S^{2n+1}, \theta_0)} = 0$, there exists a bigger $T_4 \geq T_3$ such that for all $t \geq T_4$, we have $\|f_\phi - f(\widehat{\Theta}(t))\|_{L^2(S^{2n+1}, \theta_0)} \leq \frac{\epsilon^{2n+2}}{2C_3(2C_0)^{2n+2}}$ where C_0 is given in the inequality (5.14), and $C_3 = C_1 + C_2$ where C_1 and C_2 are respectively given in the inequalities (5.12) and (5.13). Then we define $\delta = \min \left\{ \frac{\epsilon^{4n+4}}{4C_3^2(2C_0)^{4n+4}}, g(T_4) \right\} > 0$. Since $\lim_{t \rightarrow \infty} g(t) = 0$ in view of Lemma 4.1 and 4.13, there exists a $T_5 \geq T_4 + 1$ such that $g(T_5) < \delta$. Hence the set $\{t : t \geq T_4 + 1 \text{ and } g(t) < \delta\}$ is non-empty. Finally we select $T_1(u_0) \equiv T_1(\epsilon, u_0) = \inf\{t : t \geq T_4 + 1 \text{ and } g(t) < \delta\}$. We need the following two properties: (i) $T_1(u_0)$ is continuously dependent on u_0 in $S_{4n+4}^2(S^{2n+1}, \theta_0)$ and (ii) for all $t \geq T_1(u)$, $\|v-1\|_{C_P^1(S^{2n+1})} < \epsilon$.

In fact, (i) follows from monotonicity of g and continuous dependence on the initial data of our flow in $S_{4n+4}^2(S^{2n+1}, \theta_0)$ -norm as we did in Lemma 5.2. For (ii), observe that if $t > T_1(u_0)$, then $g(t) < g(T_1(u_0)) \leq \delta$ thanks to the fact that g is decreasing. Since $E_f(u)(t) - R_{\theta_0} \text{Vol}(S^{2n+1}, \theta_0)^{\frac{1}{n+1}} f(Q)^{-\frac{n}{n+1}} \geq 0$ for all $t \geq 0$ in view of Proposition 2.1 and Lemma 4.13, we conclude that $F_2(t) \leq \delta$ for all $t \geq T_1(u_0)$. Thus by estimates (5.12), (5.13) and (5.14), if $t > T_1(u_0) > T_4$, then

we have

$$\begin{aligned} \|v - 1\|_{C_P^1(S^{2n+1})} &\leq C_0 \left[(C_1 + C_2) (F_2(t)^{\frac{1}{2}} + \|f_\phi - f(\widehat{\Theta}(t))\|_{L^2(S^{2n+1}, \theta_0)}) \right]^{\frac{1}{2n+2}} \\ &\leq C_0 \left[(C_1 + C_2) \left(\delta^{\frac{1}{2}} + \frac{\epsilon^{2n+2}}{2C_3(2C_0)^{2n+2}} \right) \right]^{\frac{1}{2n+2}} < \epsilon. \end{aligned}$$

Therefore our claim is established.

Then we choose two positive constants σ_1, σ_2 to normalize the two functions $v = u(T_1) \circ \phi(\det(d\phi))^{\frac{n}{2n+2}}$ and 1, such that

$$(5.16) \quad \sigma_1^{2+\frac{2}{n}} \int_{S^{2n+1}} f \circ \phi v^{2+\frac{2}{n}} dV_{\theta_0} = 1 \quad \text{and} \quad \sigma_2^{2+\frac{2}{n}} \int_{S^{2n+1}} f \circ \phi dV_{\theta_0} = 1.$$

By (5.10), we have

$$(5.17) \quad |\sigma_1 - \sigma_2| = O(\epsilon).$$

Now we define a homotopy on L_{β_0} by

$$H(s, u_0) = \begin{cases} u(3sT_1, u_0), & \text{if } 0 \leq s \leq \frac{1}{3}; \\ \frac{1}{\sigma_1} \left[(2-3s)\sigma_1^{2+\frac{2}{n}} u(T_1, u_0)^{2+\frac{2}{n}} + (3s-1)\sigma_2^{2+\frac{2}{n}} \det(d\phi^{-1}) \right]^{\frac{n}{2n+2}} & \text{if } \frac{1}{3} \leq s \leq \frac{2}{3}; \\ \frac{\sigma_2}{\sigma_1} \left[\det(d(\Psi \circ \delta_{-q(T_1), 3(1-s)r(T_1)+(3s-2)} \circ \pi)) \right]^{\frac{n}{2n+2}}, & \text{if } \frac{2}{3} \leq s \leq 1. \end{cases}$$

Obviously, $H(s, u_0)$ induces a contraction within C_*^∞ . One calculates that $E_f(H(s, u_0)) \leq \beta_0$ if $s \in [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Hence we have left to check that we also have $E_f(H(s, u_0)) \leq \beta_0$ for $s \in [\frac{1}{3}, \frac{2}{3}]$. To do this, for simplicity, set $F(s) = E_f(H(s, u_0))$ for $s \in [\frac{1}{3}, \frac{2}{3}]$. Then we claim that for sufficiently large $T_1 > 0$, there holds

$$(5.18) \quad \frac{d^2}{ds^2} F(s) > 0 \quad \text{for all } s \in [\frac{1}{3}, \frac{2}{3}].$$

Thus we can conclude that $F(s)$ achieves its maximum value at $s = \frac{1}{3}$ or $s = \frac{2}{3}$, namely,

$$E_f(H(s, u_0)) \leq \max \left\{ E_f(H(\frac{1}{3}, u_0)), E_f(H(\frac{2}{3}, u_0)) \right\} \leq \beta_0 \quad \text{for all } s \in [\frac{1}{3}, \frac{2}{3}].$$

So the homotopy $H(s, u_0)$ is essentially a contraction within C_f^∞ .

In order to show (5.18), first by conformal invariance of the energy, we have

$$E_f(H(s, u_0)) = E_{f \circ \phi} \left(H(s, u_0) \circ \phi(\det(d\phi))^{\frac{n}{2n+2}} \right).$$

Then if we set

$$(5.19) \quad v_s^{2+\frac{2}{n}} = (2-3s)(\sigma_1 v)^{2+\frac{2}{n}} + (3s-1)\sigma_2^{2+\frac{2}{n}},$$

we have

$$\sigma_1 H(s, u_0) \circ \phi(\det(d\phi))^{\frac{n}{2n+2}} = v_s$$

and

$$E_f(H(s, u_0)) = E_{f \circ \phi}(v_s)$$

by using the fact that $E_f(\sigma u) = E_f(u)$ for any constant $\sigma > 0$. Hence we only need to estimate the energy $E_{f \circ \phi}(v_s)$ for $s \in [\frac{1}{3}, \frac{2}{3}]$. Now we denote a dot the s -derivative. We can derive from (5.19) that

$$(5.20) \quad \dot{v}_s = 3v_s^{-1-\frac{2}{n}}(\sigma_2^{2+\frac{2}{n}} - (\sigma_1 v)^{2+\frac{2}{n}})/(2 + \frac{2}{n}).$$

One has the estimate

$$(5.21) \quad \|\dot{v}_s\|_{C^0(S^{2n+1})} = O(\epsilon)$$

thanks to (5.10) and (5.17). Note also that by (5.20) we have

$$(5.22) \quad \ddot{v}_s = -(1 + \frac{2}{n})v_s^{-1}(\dot{v}_s)^2.$$

By (5.10) and (5.17), we have

$$\begin{aligned} \|v_s^{2+\frac{2}{n}} - \sigma_1^{2+\frac{2}{n}}\|_{C^0} &= \|(2-3s)(\sigma_1 v)^{2+\frac{2}{n}} + (3s-1)\sigma_2^{2+\frac{2}{n}} - \sigma_1^{2+\frac{2}{n}}\|_{C^0} \\ &= \|(2-3s)\sigma_1^{2+\frac{2}{n}}(v^{2+\frac{2}{n}} - 1) + (3s-1)(\sigma_2^{2+\frac{2}{n}} - \sigma_1^{2+\frac{2}{n}})\|_{C^0} \\ &= O(\epsilon), \end{aligned}$$

which implies that

$$(5.23) \quad \|v_s - \sigma_1\|_{C^0} = O(\epsilon).$$

It follows from (5.19) that

$$(2 + \frac{2}{n})v_s^{1+\frac{2}{n}}\nabla_{\theta_0}v_s = (2-3s)(2 + \frac{2}{n})\sigma_1^{2+\frac{2}{n}}v^{1+\frac{2}{n}}\nabla_{\theta_0}v,$$

which implies that

$$(5.24) \quad \|\nabla_{\theta_0}v_s\|_{C^0} = O(\epsilon)$$

by (5.10). Moreover, it follows from (2.9), (5.16) and (5.19) that

$$\begin{aligned} &\int_{S^{2n+1}} f \circ \phi v_s^{2+\frac{2}{n}} dV_{\theta_0} \\ (5.25) \quad &= (2-3s)\sigma_1^{2+\frac{2}{n}} \int_{S^{2n+1}} f \circ \phi v^{2+\frac{2}{n}} dV_{\theta_0} + (3s-1)\sigma_2^{2+\frac{2}{n}} \int_{S^{2n+1}} f \circ \phi dV_{\theta_0} \\ &= (2-3s) + (3s-1) = 1 \end{aligned}$$

and

$$\begin{aligned} (5.26) \quad &\int_{S^{2n+1}} (x, \bar{x}) v_s^{2+\frac{2}{n}} dV_{\theta_0} \\ &= (2-3s)\sigma_1^{2+\frac{2}{n}} \int_{S^{2n+1}} (x, \bar{x}) v^{2+\frac{2}{n}} dV_{\theta_0} + (3s-1)\sigma_2^{2+\frac{2}{n}} \int_{S^{2n+1}} (x, \bar{x}) dV_{\theta_0} = 0. \end{aligned}$$

From (5.25) and (5.26), we obtain

$$(5.27) \quad \int_{S^{2n+1}} f \circ \phi v_s^{1+\frac{2}{n}} \dot{v}_s dV_{\theta_0} = 0 \quad \text{and} \quad \int_{S^{2n+1}} (x, \bar{x}) v_s^{1+\frac{2}{n}} \dot{v}_s dV_{\theta_0} = 0.$$

On the other hand, for any positive function f , a direct computation yields

$$\begin{aligned}
 (5.28) \quad dE_f(u)(\eta) &= \frac{d}{dr} E_f(u + r\eta) \Big|_{r=0} \\
 &= 2 \left(\int_{S^{2n+1}} f u^{2+\frac{2}{n}} dV_{\theta_0} \right)^{-\frac{n}{n+1}} \\
 &\quad \cdot \left[\int_{S^{2n+1}} \left(\left(2 + \frac{2}{n} \right) \langle \nabla_{\theta_0} u, \nabla_{\theta_0} \eta \rangle_{\theta_0} + R_{\theta_0} u \eta \right) dV_{\theta_0} \right. \\
 &\quad \left. - \frac{\int_{S^{2n+1}} \left(\left(2 + \frac{2}{n} \right) |\nabla_{\theta_0} u|_{\theta_0}^2 + R_{\theta_0} u^2 \right) dV_{\theta_0}}{\int_{S^{2n+1}} f u^{2+\frac{2}{n}} dV_{\theta_0}} \int_{S^{2n+1}} f u^{1+\frac{2}{n}} \eta dV_{\theta_0} \right]
 \end{aligned}$$

and

$$\begin{aligned}
 (5.29) \quad d^2 E_f(u)(\zeta, \eta) &= \frac{d}{dr} [dE_f(u + r\zeta)(\eta)] \Big|_{r=0} \\
 &= 2 \left(\int_{S^{2n+1}} f u^{2+\frac{2}{n}} dV_{\theta_0} \right)^{-\frac{2n+1}{n+1}} \\
 &\quad \cdot \left\{ \int_{S^{2n+1}} \left(\left(2 + \frac{2}{n} \right) \langle \nabla_{\theta_0} \zeta, \nabla_{\theta_0} \eta \rangle_{\theta_0} + R_{\theta_0} \zeta \eta \right) dV_{\theta_0} \cdot \int_{S^{2n+1}} f u^{2+\frac{2}{n}} dV_{\theta_0} \right. \\
 &\quad - \left(1 + \frac{2}{n} \right) \int_{S^{2n+1}} \left(\left(2 + \frac{2}{n} \right) |\nabla_{\theta_0} u|_{\theta_0}^2 + R_{\theta_0} u^2 \right) dV_{\theta_0} \cdot \int_{S^{2n+1}} f u^{\frac{2}{n}} \zeta \eta dV_{\theta_0} \\
 &\quad - 2 \left[\int_{S^{2n+1}} \left(\left(2 + \frac{2}{n} \right) \langle \nabla_{\theta_0} u, \nabla_{\theta_0} \zeta \rangle_{\theta_0} + R_{\theta_0} u \zeta \right) dV_{\theta_0} \cdot \int_{S^{2n+1}} f u^{1+\frac{2}{n}} \eta dV_{\theta_0} \right. \\
 &\quad \left. + \int_{S^{2n+1}} \left(\left(2 + \frac{2}{n} \right) \langle \nabla_{\theta_0} u, \nabla_{\theta_0} \eta \rangle_{\theta_0} + R_{\theta_0} u \eta \right) dV_{\theta_0} \cdot \int_{S^{2n+1}} f u^{1+\frac{2}{n}} \zeta dV_{\theta_0} \right] \\
 &\quad \left. + \left(2 + \frac{2}{n} \right) \frac{\int_{S^{2n+1}} \left(\left(2 + \frac{2}{n} \right) |\nabla_{\theta_0} u|_{\theta_0}^2 + R_{\theta_0} u^2 \right) dV_{\theta_0}}{\int_{S^{2n+1}} f u^{2+\frac{2}{n}} dV_{\theta_0}} \cdot \int_{S^{2n+1}} f u^{1+\frac{2}{n}} \zeta dV_{\theta_0} \cdot \int_{S^{2n+1}} f u^{1+\frac{2}{n}} \eta dV_{\theta_0} \right\}.
 \end{aligned}$$

We observe that Folland-Stein embedding theorem shows that the map

$$u \mapsto d^2 E_f(u)(\cdot, \cdot) \in L(S_1^2(S^{2n+1}, \theta_0) \times S_1^2(S^{2n+1}, \theta_0), \mathbb{R})$$

is continuous.

Notice that

$$\begin{aligned}
 &\int_{S^{2n+1}} \langle \nabla_{\theta_0} v_s, \nabla_{\theta_0} (v_s^{-1} \dot{v}_s^2) \rangle_{\theta_0} dV_{\theta_0} \\
 &= - \int_{S^{2n+1}} v_s^{-2} |\nabla_{\theta_0} v_s|_{\theta_0}^2 \dot{v}_s^2 dV_{\theta_0} + 2 \int_{S^{2n+1}} v_s^{-1} \dot{v}_s \langle \nabla_{\theta_0} v_s, \nabla_{\theta_0} \dot{v}_s \rangle_{\theta_0} dV_{\theta_0} \\
 &= O(\|\nabla_{\theta_0} v_s\|_{C^0}^2) (\|\dot{v}_s\|_{L^2}^2 + \|\nabla_{\theta_0} \dot{v}_s\|_{L^2}^2) \\
 &= O(\epsilon) (\|\dot{v}_s\|_{L^2}^2 + \|\nabla_{\theta_0} \dot{v}_s\|_{L^2}^2)
 \end{aligned}$$

by (5.23) and (5.24). Using (5.21)-(5.25), (5.28) and (5.29), we obtain

$$\begin{aligned}
(5.30) \quad & \frac{d^2}{ds^2} E_{f \circ \phi}(v_s) \\
&= d^2 E_{f \circ \phi}(v_s)(\dot{v}_s, \dot{v}_s) + dE_{f \circ \phi}(v_s)(\ddot{v}_s) \\
&= 2\left(2 + \frac{2}{n}\right) \int_{S^{2n+1}} |\nabla_{\theta_0} \dot{v}_s|_{\theta_0}^2 dV_{\theta_0} + 2R_{\theta_0} \int_{S^{2n+1}} \dot{v}_s^2 dV_{\theta_0} \\
&\quad - 2\left(1 + \frac{2}{n}\right) \int_{S^{2n+1}} \left(\left(2 + \frac{2}{n}\right) |\nabla_{\theta_0} v_s|_{\theta_0}^2 + R_{\theta_0} v_s^2 \right) dV_{\theta_0} \cdot \int_{S^{2n+1}} f v_s^{\frac{2}{n}} \dot{v}_s^2 dV_{\theta_0} \\
&\quad - 8 \int_{S^{2n+1}} \left(\left(2 + \frac{2}{n}\right) \langle \nabla_{\theta_0} v_s, \nabla_{\theta_0} \dot{v}_s \rangle_{\theta_0} + R_{\theta_0} v_s \dot{v}_s \right) dV_{\theta_0} \cdot \int_{S^{2n+1}} f v_s^{1+\frac{2}{n}} \dot{v}_s dV_{\theta_0} \\
&\quad + 2\left(2 + \frac{2}{n}\right) \int_{S^{2n+1}} \left(\left(2 + \frac{2}{n}\right) |\nabla_{\theta_0} v_s|_{\theta_0}^2 + R_{\theta_0} v_s^2 \right) dV_{\theta_0} \cdot \left(\int_{S^{2n+1}} f v_s^{1+\frac{2}{n}} \dot{v}_s dV_{\theta_0} \right)^2 \\
&\quad + 2 \int_{S^{2n+1}} \left(\left(2 + \frac{2}{n}\right) \langle \nabla_{\theta_0} v_s, \nabla_{\theta_0} \ddot{v}_s \rangle_{\theta_0} + R_{\theta_0} v_s \ddot{v}_s \right) dV_{\theta_0} \\
&\quad - 2 \int_{S^{2n+1}} \left(\left(2 + \frac{2}{n}\right) |\nabla_{\theta_0} v_s|_{\theta_0}^2 + R_{\theta_0} v_s^2 \right) dV_{\theta_0} \cdot \int_{S^{2n+1}} f v_s^{1+\frac{2}{n}} \ddot{v}_s dV_{\theta_0} \\
&= \left(2\left(2 + \frac{2}{n}\right) + O(\epsilon) \right) \int_{S^{2n+1}} |\nabla_{\theta_0} \dot{v}_s|_{\theta_0}^2 dV_{\theta_0} - \left(\frac{4R_{\theta_0}}{n} + O(\epsilon) \right) \int_{S^{2n+1}} \dot{v}_s^2 dV_{\theta_0}.
\end{aligned}$$

Now we decompose $\dot{v}_s = \varphi + w$, where

$$w = \int_{S^{2n+1}} \dot{v}_s dV_{\theta_0} + \sum_{i=1}^{2n+2} \left(\int_{S^{2n+1}} \dot{v}_s \varphi_i dV_{\theta_0} \right) \varphi_i$$

and $\{\varphi_i\}$ are the eigenfunctions of $-\Delta_{\theta_0}$ given in section 4.1. Let $\widehat{P}(t)$ be the limit point of the conformal CR diffeomorphism $\phi(t)$ in view of Lemma 2.3, one finds by (5.27) that

$$\begin{aligned}
& \sigma_1^{1+\frac{2}{n}} f(\widehat{P}(t)) \int_{S^{2n+1}} \dot{v}_s dV_{\theta_0} \\
&= \int_{S^{2n+1}} (\sigma_1^{1+\frac{2}{n}} f(\widehat{P}(t)) - f \circ \phi v_s^{1+\frac{2}{n}}) \dot{v}_s dV_{\theta_0} \\
&= \int_{S^{2n+1}} \left[\sigma_1^{1+\frac{2}{n}} (f(\widehat{P}(t)) - f \circ \phi) + f \circ \phi (\sigma_1^{1+\frac{2}{n}} - v_s^{1+\frac{2}{n}}) \right] \dot{v}_s dV_{\theta_0}
\end{aligned}$$

and

$$\sigma_1^{1+\frac{2}{n}} \int_{S^{2n+1}} \dot{v}_s \varphi_i dV_{\theta_0} = \int_{S^{2n+1}} (\sigma_1^{1+\frac{2}{n}} - v_s^{1+\frac{2}{n}}) \dot{v}_s \varphi_i dV_{\theta_0}.$$

Hence from Lemma 2.3, (5.23), and Hölder's inequality, we obtain

$$\int_{S^{2n+1}} \dot{v}_s dV_{\theta_0} = o(1) \|\dot{v}_s\|_{L^2} \quad \text{and} \quad \int_{S^{2n+1}} \dot{v}_s \varphi_i dV_{\theta_0} = O(\epsilon) \|\dot{v}_s\|_{L^2},$$

which implies that

$$\begin{aligned}
& \|w\|_{L^2} = o(1) \|\dot{v}_s\|_{L^2} \quad \text{and} \\
(5.31) \quad & \|\nabla_{\theta_0} w\|_{L^2} = \left\| \sum_{i=1}^{2n+2} \left(\int_{S^{2n+1}} \dot{v}_s \varphi_i dV_{\theta_0} \right) \nabla_{\theta_0} \varphi_i \right\|_{L^2} = O(\epsilon) \|\dot{v}_s\|_{L^2}
\end{aligned}$$

by the definition of w . By (5.31) and $\dot{v}_s = \varphi + w$, we have

$$\begin{aligned}
 (1 + o(1)) \int_{S^{2n+1}} |\nabla_{\theta_0} \dot{v}_s|_{\theta_0}^2 dV_{\theta_0} &= \int_{S^{2n+1}} |\nabla_{\theta_0} \varphi|_{\theta_0}^2 dV_{\theta_0} \\
 (5.32) \qquad \qquad \qquad &\geq \lambda_{2n+3} \int_{S^{2n+1}} \varphi^2 dV_{\theta_0} \\
 &= (\lambda_{2n+3} + o(1)) \int_{S^{2n+1}} \dot{v}_s^2 dV_{\theta_0}.
 \end{aligned}$$

Combining (5.30) and (5.32), we can conclude that

$$\begin{aligned}
 &\frac{d^2}{ds^2} E_{f \circ \phi}(v_s) \\
 &= \left(2\left(2 + \frac{2}{n}\right) + O(\epsilon)\right) \int_{S^{2n+1}} |\nabla_{\theta_0} \dot{v}_s|_{\theta_0}^2 dV_{\theta_0} - \left(\frac{4R_{\theta_0}}{n} + O(\epsilon)\right) \int_{S^{2n+1}} \dot{v}_s^2 dV_{\theta_0} \\
 &\geq \left(2\left(2 + \frac{2}{n}\right)\lambda_{2n+3} - \frac{4R_{\theta_0}}{n} + o(1)\right) \int_{S^{2n+1}} \dot{v}_s^2 dV_{\theta_0} > 0,
 \end{aligned}$$

since

$$2\left(2 + \frac{2}{n}\right)\lambda_{2n+3} - \frac{4R_{\theta_0}}{n} = 2\left(2 + \frac{2}{n}\right)\lambda_{2n+3} - \frac{4}{n} \cdot \frac{n(n+1)}{2} > 0$$

thanks to $\lambda_{2n+3} > n/2$.

So we have established (5.18). Notice that our homotopy $H(s, u_0)$ is the one which is homotopic to the constant σ_2/σ_1 . Since this is a constant, its energy is always less than β_0 . Then clearly this constant will be homotopic to the constant 1 in the set C_f^∞ . So we have finished the proof of (i). \square

Proof of Proposition 5.1 (ii). In order to prove (ii), we re-scale the time t by letting $\tau(t)$ solve

$$(5.33) \qquad \frac{d\tau}{dt} = \min \left\{ \frac{1}{2}, \epsilon^2(t, u_0) \right\}, \quad \tau(0) = 0.$$

By (4.82), we see that $\tau(t) \rightarrow \infty$ as $t \rightarrow \infty$. Set $U = u(\tau(t), u_0)$ and $\Gamma(\tau) = \Theta(\tau(t), u_0)$. As in the proof of Proposition 4.12, we have

$$\frac{d}{d\tau} (\text{Vol}(S^{2n+1}, \theta_0)^2 - |\Gamma(\tau)|^2) = 4(n+1)A_2A_5\alpha\epsilon^3(\Delta_{\theta_0}f(\widehat{\Gamma(\tau)}) + O(1)|\nabla_{\theta_0}f(\widehat{\Gamma(\tau)})|_{\theta_0}^2 + O(\epsilon)).$$

and

$$(5.34) \qquad \left| \frac{d}{d\tau} (\text{Vol}(S^{2n+1}, \theta_0)^2 - |\Gamma(\tau)|^2) \right| \leq C(\text{Vol}(S^{2n+1}, \theta_0)^2 - |\Gamma(\tau)|^2)^2,$$

with error $O(1)$ which means it is bounded as $\epsilon \rightarrow 0$. In the following argument, we will still use t for $\tau(t)$, $u(t, u_0)$ for $U(\tau(t), u_0)$ and $\Theta(t)$ for $\Gamma(\tau(t))$ when there is no confusion arising.

Thus, for a given $0 < \nu \leq \nu_0$, we claim that there exists $T > 0$ such that $u(T, L_{\beta_i - \nu}) \subset L_{\beta_{i+1} + \nu}$. Suppose on the contrary, there exist, for each integer k , $T_k \geq 2k$ and an initial data $u_k \in L_{\beta_i - \nu} \setminus L_{\beta_{i+1} + \nu}$, such that

$$E_f(u(T_k, u_k)) > \beta_{i+1} + \nu \quad \text{for all } k.$$

By Lemma 4.1, there exists a sequence $t_k \in [T_k/2, T_k]$ such that $\int_{S^{2n+1}} |\alpha(t_k)f - R_{\theta_k}|^2 dV_{\theta_k} \rightarrow 0$ as $k \rightarrow \infty$, where $\theta_k = u(t_k, u_k)^{\frac{2}{n}}\theta_0$, $k \in \mathbb{Z}^+$ and R_{θ_k} is the Webster

scalar curvature of θ_k . In fact, if for all $t \in [T_k/2, T_k]$, $\int_{S^{2n+1}} |\alpha(t)f - R_{\theta(t)}|^2 dV_{\theta(t)} \geq \epsilon_0 > 0$ for some fixed $\epsilon_0 > 0$ and k sufficiently large, we would have

$$\beta_i - \beta_{i+1} - 2\nu \geq \int_{\frac{T_k}{2}}^{T_k} \left(-\frac{dE_f(u(t, u_k))}{dt} \right) dt \geq \epsilon_0 C T_k / 2$$

which contradicts the assumption that $T_k \rightarrow \infty$ as $k \rightarrow \infty$.

Let $\widehat{\Theta}(t_k, u_k) \in S^{2n+1}$ be the shadow points of the flow with the initial data u_k valued at time t_k . And $v_k = u(t_k, u_k) \circ \phi(t_k)(\det(d\phi(t_k)))^{\frac{n}{2n+2}}$. Then $v_k \rightarrow 1$ as k sufficiently large. Up to a subsequence, the limit $\widehat{\Theta} = \lim_{k \rightarrow \infty} \widehat{\Theta}(t_k, u_k)$ exists and $v_k \rightarrow 1$ in $C^{1,\alpha}$ for some $\alpha > 0$ according to Lemma 2.3. Then $\widehat{\Theta}$ must be a critical point of f by (5.34). Then, as in the proof of Lemma 4.13, the direct calculation shows that $E_f(u(t_k, u_k)) \rightarrow R_{\theta_0} \text{Vol}(S^{2n+1}, \theta_0)^{\frac{1}{n+1}} f(\widehat{\Theta})^{-\frac{n}{n+1}}$ as $k \rightarrow \infty$. Since $E_f(u_k) \leq \beta_i - \nu$, by Proposition 2.1, there hold $\widehat{\Theta} = p_{i_0}$ for some $i_0 > i$ and

$$\begin{aligned} E_f(u(T_k, u_k)) &\leq E_f(u(t_k, u_k)) = E_{f \circ \phi(t_k)}(v(t_k, u_k)) \\ &\leq R_{\theta_0} \text{Vol}(S^{2n+1}, \theta_0)^{\frac{1}{n+1}} f(\widehat{\Theta})^{-\frac{n}{n+1}} + \nu \leq \beta_{i+1} + \nu, \end{aligned}$$

which yields a contradiction.

For $u_0 \in L_{\beta_i - \nu} \setminus L_{\beta_{i+1} + \nu}$, let

$$T(u_0) = \inf\{t \geq 0 : E_f(u(t, u_0)) \leq \beta_{i+1} + \nu\} \leq T.$$

As in (i), $T(u_0)$ continuously depends on u_0 and the map $K(s, u_0) = u(sT(u_0), u_0)$ for $0 \leq s \leq 1$ if $u \in L_{\beta_i - \nu} \setminus L_{\beta_{i+1} + \nu}$ and $K(s, u_0) = u_0$ if $u_0 \in L_{\beta_{i+1} + \nu}$ defines the desired homotopy equivalence between $L_{\beta_{i+1} + \nu}$ and $L_{\beta_i - \nu}$. This finishes the proof of (ii). \square

For the proof of (iii) and (iv), we need some additional lemmas.

Lemma 5.3. *With two dimensional constants $C_1 > 0$, $C_2 > 0$, provided that $\|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)}$ is sufficiently small, there holds*

$$C_1 \|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)}^2 \geq E(v) - R_{\theta_0} \text{Vol}(S^{2n+1}, \theta_0) \geq C_2 \|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)}^2,$$

for all $v \in S_1^2(S^{2n+1}, \theta_0) \cap C_*^\infty$, the conformal factor of the normalized contact form h satisfying (2.9).

Proof. Note that

$$E(v) - R_{\theta_0} \text{Vol}(S^{2n+1}, \theta_0) = \int_{S^{2n+1}} \left(\left(2 + \frac{2}{n}\right) |\nabla_{\theta_0} v|_{\theta_0}^2 + R_{\theta_0}(v^2 - 1) \right) dV_{\theta_0}.$$

Note also that

$$\begin{aligned} R_{\theta_0} \int_{S^{2n+1}} (v^2 - 1) dV_{\theta_0} &= R_{\theta_0} \int_{S^{2n+1}} (v - 1)^2 dV_{\theta_0} + 2R_{\theta_0} \int_{S^{2n+1}} (v - 1) dV_{\theta_0} \\ &= R_{\theta_0} \int_{S^{2n+1}} (v - 1)^2 dV_{\theta_0} + o(1) \|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)}^2. \end{aligned}$$

Thus, it is easy to derive from the above inequalities that there exists some constant $C_1 > 0$ such that

$$E(v) - R_{\theta_0} \text{Vol}(S^{2n+1}, \theta_0) \leq C_1 \|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)}^2.$$

On the other hand, let us assume that $\|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)} \leq 1$. Since v satisfies (2.9), we use (4.15) to estimate

$$\begin{aligned}
& E(v) - R_{\theta_0} \text{Vol}(S^{2n+1}, \theta_0) \\
&= (2 + \frac{2}{n}) \int_{S^{2n+1}} |\nabla_{\theta_0} v|_{\theta_0}^2 dV_{\theta_0} + R_{\theta_0} \int_{S^{2n+1}} (v - 1)^2 dV_{\theta_0} + 2R_{\theta_0} \int_{S^{2n+1}} (v - 1) dV_{\theta_0} \\
&= \min \left\{ (2 + \frac{2}{n}), R_{\theta_0} \right\} \int_{S^{2n+1}} (|\nabla_{\theta_0} v|_{\theta_0}^2 + (v - 1)^2) dV_{\theta_0} + o(1) \|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)}^2 \\
&= C_2 \int_{S^{2n+1}} (|\nabla_{\theta_0} v|_{\theta_0}^2 + (v - 1)^2) dV_{\theta_0}
\end{aligned}$$

for some constant $C_2 > 0$. □

For $r_0 > 0$ and each critical point $p_i \in S^{2n+1}$ of f , set

$$\begin{aligned}
B_{r_0}(p_i) &= \left\{ u \in C_*^\infty : \theta = u^{\frac{2}{n}} \theta_0 \text{ induces normalized contact form} \right. \\
&\quad \left. h = \phi^* \theta = v^{\frac{2}{n}} \theta_0 \text{ with } \phi = \phi_{-p, \epsilon} \text{ for some } p \in S^{2n+1} \text{ and} \right. \\
&\quad \left. 0 < \epsilon \leq 1 \text{ such that } \|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)}^2 + |p - p_i|^2 + \epsilon^2 < r_0^2 \right\}.
\end{aligned}$$

As shown in [22], the new coordinates (ϵ, p, v) are introduced to $u \in B_{r_0}(p_i)$. Under the assumption on f , from Morse lemma, we introduce the local coordinates $p = p^+ + p^-$ near $p_i = 0$, such that

$$f(p) = f(p_i) + |p^+|^2 - |p^-|^2.$$

Lemma 5.4. *For $r_0 > 0$ and $u = (\epsilon, p, v) \in B_{r_0}(p_i)$, with $o(1) \rightarrow 0$ as $r_0 \rightarrow 0$, there hold*

(a)

$$\begin{aligned}
(5.35) \quad & \int_{S^{2n+1}} f \circ \phi_{-p, \epsilon} dV_h = f(p) \text{Vol}(S^{2n+1}, \theta_0) + A_6 \epsilon^2 \Delta_{\theta_0} f(p) + O(\epsilon^4) \\
& + o(1) \epsilon \|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)},
\end{aligned}$$

where A_6 is the positive constant defined as in (5.41).

(b) There holds

$$\begin{aligned}
(5.36) \quad & \left| \frac{\partial}{\partial \epsilon} E_f(u) + \frac{n}{n+1} E(u) \left(f(p) \text{Vol}(S^{2n+1}, \theta_0) \right)^{-\frac{2n+1}{n+1}} \epsilon A_6 \Delta_{\theta_0} f(p) \right| \\
& \leq C \epsilon^2 + C(\epsilon + |p - p_i|) \|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)}.
\end{aligned}$$

In particular, if $\Delta_{\theta_0} f(p) > 0$, we have

$$\begin{aligned}
(5.37) \quad & \frac{\partial}{\partial \epsilon} E_f(u) \leq -\frac{n^2}{2} f(p)^{-\frac{2n+1}{n+1}} \text{Vol}(S^{2n+1}, \theta_0)^{-\frac{2n}{n+1}} \epsilon A_6 \Delta_{\theta_0} f(p) \\
& + C \epsilon^2 + C(\epsilon + |p - p_i|) \|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)}.
\end{aligned}$$

(c) For any $q \in T_p(S^{2n+1})$, there holds

$$\begin{aligned}
(5.38) \quad & \left| \frac{\partial E_f(u)}{\partial p} \cdot q + \frac{n}{n+1} E(v) f(p)^{-\frac{2n+1}{n+1}} df(p) \cdot q \right| \\
& \leq C \epsilon (\epsilon + \|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)}) |q|.
\end{aligned}$$

(d) *There exists a uniform constant $C_0 > 0$ such that*

$$(5.39) \quad \left\langle \frac{\partial}{\partial v} E_f(u), v - 1 \right\rangle \geq C_0 \|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)}^2 + o(1)\epsilon \|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)},$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing of $S_1^2(S^{2n+1}, \theta_0)$ with its dual.

Proof. For notational convenience, let

$$A = A(u) = \int_{S^{2n+1}} f \circ \phi_{-p, \epsilon} dV_h.$$

(a) Observe that

$$A - f(p) \text{Vol}(S^{2n+1}, \theta_0) = \int_{S^{2n+1}} (f \circ \phi_{-p, \epsilon} - f(p)) dV_{\theta_0} + I,$$

where the error term I is given by

$$I = \int_{S^{2n+1}} (f \circ \phi_{-p, \epsilon} - f(p)) (v^{2+\frac{2}{n}} - 1) dV_{\theta_0}$$

which can be estimated as follows:

$$(5.40) \quad \begin{aligned} |I| &\leq \|f \circ \phi_{-p, \epsilon} - f(p)\|_{L^2(S^{2n+1}, \theta_0)} \|v^{2+\frac{2}{n}} - 1\|_{L^2(S^{2n+1}, \theta_0)} \\ &\leq o(1)\epsilon \|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)} \end{aligned}$$

in view of (4.34) and $|\nabla_{\theta_0} f(p)|_{\theta_0} \rightarrow 0$ as $r_0 \rightarrow 0$. Using the expansion of f in (4.32) around p , we obtain by symmetry, (4.33) and (4.36) that

$$\begin{aligned} A - f(p) \text{Vol}(S^{2n+1}, \theta_0) &= \int_{B_{\epsilon^{-1}}(0)} (f \circ \phi_{-p, \epsilon} - f(p)) \frac{4^{n+1} dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} + O(\epsilon^{2n}) + o(1)\epsilon \|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)} \\ &= \frac{1}{2n} \epsilon^2 \Delta_{\theta_0} f(p) \int_{\mathbb{H}^n} \frac{4^{n+1} |z|^2 dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} + C \epsilon^4 \int_{\mathbb{H}^n} \frac{4^{n+1} \tau^2 dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} \\ &\quad + C \int_{B_{\epsilon^{-1}}(0)} \frac{\epsilon^3 (|z|^4 + \tau^2)^{\frac{3}{4}} dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} + O(\epsilon^{2n}) + o(1)\epsilon \|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)} \\ &= \epsilon^2 A_6 \Delta_{\theta_0} f(p) + O(\epsilon^4) + o(1)\epsilon \|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)} \end{aligned}$$

where A_6 is given by

$$(5.41) \quad A_6 := \frac{1}{2n} \int_{\mathbb{H}^n} \frac{4^{n+1} |z|^2 dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}}.$$

This yields the first assertion.

(b) Note that

$$E_f(u) = \frac{E(u)}{(\int_{S^{2n+1}} f u^{2+\frac{2}{n}} dV_{\theta_0})^{\frac{n}{n+1}}} = \frac{E(v)}{(\int_{S^{2n+1}} f \circ \phi_{-p, \epsilon} v^{2+\frac{2}{n}} dV_{\theta_0})^{\frac{n}{n+1}}}.$$

Thus it follows that

$$\frac{\partial}{\partial \epsilon} E_f(u) = -\frac{n}{n+1} E(v) A^{-\frac{2n+1}{n+1}} \frac{\partial}{\partial \epsilon} \int_{S^{2n+1}} f \circ \phi_{-p, \epsilon} dV_h.$$

Denote $\phi_{-p,\epsilon}$ by Ψ_ϵ , as in (a), we have

$$\begin{aligned} \frac{\partial}{\partial \epsilon} \int_{S^{2n+1}} f \circ \phi_{-p,\epsilon} dV_h &= \int_{\mathbb{H}^n} \frac{\partial}{\partial \epsilon} f(\Psi_\epsilon(z, \tau)) \frac{4^{n+1} dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} \\ &\quad + \int_{\mathbb{H}^n} \frac{\partial}{\partial \epsilon} f(\Psi_\epsilon(z, \tau)) (v^{2+\frac{2}{n}} - 1) \frac{4^{n+1} dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} \\ &= I + II. \end{aligned}$$

First note that

$$\begin{aligned} \frac{\partial}{\partial \epsilon} f(\Psi_\epsilon(z, \tau)) &= \sum_{i=1}^n \frac{\partial f}{\partial x_i} \circ \Psi_\epsilon(z, \tau) \frac{2z_i(1 - \epsilon^2|z|^2 + \sqrt{-1}\epsilon^2\tau)}{(1 + \epsilon^2|z|^2 - \sqrt{-1}\epsilon^2\tau)^2} \\ &\quad + \frac{\partial f}{\partial x_{n+1}} \circ \Psi_\epsilon(z, \tau) \frac{-4\epsilon(|z|^2 - \sqrt{-1}\tau)}{(1 + \epsilon^2|z|^2 - \sqrt{-1}\epsilon^2\tau)^2}. \end{aligned}$$

Since $\sum_{i=1}^{n+1} \left| \frac{\partial f}{\partial x_i} \right|^2 = \frac{4}{(1 + |z|^2)^2 + \tau^2} |\nabla_{\theta_0} f|_{\theta_0}^2$, we have

$$(5.42) \quad \left| \frac{\partial}{\partial \epsilon} f(\Psi_\epsilon(z, \tau)) \right| \leq C(|z|^4 + \tau^2)^{\frac{1}{4}} \quad \text{for } (z, \tau) \in \mathbb{H}^n.$$

Now using the expansion of f in (4.32), we have

$$\begin{aligned} \frac{\partial}{\partial \epsilon} f(\Psi_\epsilon(z, \tau)) &= \sum_{i=1}^n \left(\frac{\partial f(p)}{\partial a_i} a_i + \frac{\partial f(p)}{\partial b_i} b_i \right) + 2\epsilon \frac{\partial f(p)}{\partial \tau} \tau + 2\epsilon^3 \frac{\partial^2 f(p)}{\partial \tau^2} \tau^2 \\ &\quad + \epsilon \sum_{i,j=1}^n \left(\frac{\partial^2 f(p)}{\partial a_i \partial a_j} a_i a_j + \frac{\partial^2 f(p)}{\partial b_i \partial b_j} b_i b_j \right) \\ &\quad + \frac{3}{2}\epsilon^2 \sum_{i=1}^n \left(\frac{\partial^2 f(p)}{\partial a_i \partial \tau} a_i \tau + \frac{\partial^2 f(p)}{\partial b_i \partial \tau} b_i \tau \right) + O(\epsilon^2(|z|^4 + \tau^2)^{\frac{3}{4}}) \end{aligned}$$

in $B_{\epsilon^{-1}}(0)$. By symmetry, we obtain

$$\begin{aligned} I &= \frac{\epsilon}{2n} \Delta_{\theta_0} f(p) \int_{\mathbb{H}^n} \frac{4^{n+1} |z|^2 dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} + O(\epsilon^{2n}) \\ &\quad + C\epsilon^3 \int_{B_{\epsilon^{-1}}(0)} \frac{\tau^2 dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} + C\epsilon^2 \int_{B_{\epsilon^{-1}}(0)} \frac{(|z|^4 + \tau^2)^{\frac{3}{4}} dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} \\ &= \epsilon A_6 \Delta_{\theta_0} f(p) + O(\epsilon^2) \end{aligned}$$

where A_6 is the constant defined in (5.41). On the other hand, the expansion of f in $B_{\epsilon^{-1}}(0)$ to the first order

$$\begin{aligned} \frac{\partial}{\partial \epsilon} f(\Psi_\epsilon(z, \tau)) &= \sum_{j=1}^n \left(\frac{\partial f(p)}{\partial a_j} a_j + \frac{\partial f(p)}{\partial b_j} b_j \right) + O(\epsilon(|z|^4 + \tau^2)^{\frac{1}{4}}) \\ &= \sum_{j=1}^n \left[\left(\frac{\partial f(p)}{\partial a_j} - \frac{\partial f(p_i)}{\partial a_j} \right) a_j + \left(\frac{\partial f(p)}{\partial b_j} - \frac{\partial f(p_i)}{\partial b_j} \right) b_j \right] + O(\epsilon \sqrt{|z|^4 + \tau^2}) \end{aligned}$$

gives the uniform estimate

$$(5.43) \quad \left| \frac{\partial}{\partial \epsilon} f(\Psi_\epsilon(z, \tau)) \right| \leq C|p - p_i||z| + C\epsilon \sqrt{|z|^4 + \tau^2} \quad \text{in } B_{\epsilon^{-1}}(0).$$

By (5.42) and (5.43), we get the estimate

$$\begin{aligned} |II| &\leq C \left[(\epsilon + |p - p_i|) \int_{B_{\epsilon^{-1}}(0)} |v^{2+\frac{2}{n}} - 1| \frac{(1 + \sqrt{|z|^4 + \tau^2}) dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} \right. \\ &\quad \left. + \int_{\mathbb{H}^n \setminus B_{\epsilon^{-1}}(0)} |v^{2+\frac{2}{n}} - 1| \frac{(|z|^4 + \tau^2)^{\frac{1}{4}} dz d\tau}{(\tau^2 + (1 + |z|^2)^2)^{n+1}} \right] \\ &\leq C(\epsilon + |p - p_i|) \|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)} \end{aligned}$$

by (4.1) and (4.2). Thus, (5.36) follows from the estimates above and (a). Moreover, if $\Delta_{\theta_0} f(p) > 0$, the lower bound of $E(u) = E(v) \geq R_{\theta_0} \text{Vol}(S^{2n+1}, \theta_0)^{\frac{1}{n+1}} = \frac{n(n+1)}{2} \text{Vol}(S^{2n+1}, \theta_0)^{\frac{1}{n+1}}$ in view of Lemma 2.3 in [17] and $u \in B_{r_0}(p_i) \subset C_*^\infty$, together with (5.36) derive the estimate (5.37).

(c) For any $q \in T_p(S^{2n+1})$, as shown in (4.32), we obtain the expansion of $d(f \circ \phi_{-p, \epsilon}) \cdot q$ around p as

$$\begin{aligned} &d(f \circ \phi_{-p, \epsilon}) \cdot q - df(p) \cdot q \\ &= \frac{\epsilon}{2} \sum_{i,j=1}^n \left(\frac{\partial^2 f(p)}{\partial a_i \partial a_j} \Big|_{(z, \tau)=(0,0)} a_i \tilde{a}_j + \frac{\partial^2 f(p)}{\partial b_i \partial b_j} \Big|_{(z, \tau)=(0,0)} b_i \tilde{b}_j \right) \\ &\quad + \frac{1}{2} \sum_{i=1}^n \left(\frac{\partial^2 f(p)}{\partial a_i \partial \tau} \Big|_{(z, \tau)=(0,0)} (\epsilon \tilde{a}_i \tau + \epsilon^2 a_i \tilde{\tau}) + \frac{\partial^2 f(p)}{\partial b_i \partial \tau} \Big|_{(z, \tau)=(0,0)} (\epsilon \tilde{b}_i \tau + \epsilon^2 b_i \tilde{\tau}) \right) \\ &\quad + \frac{\epsilon^2}{2} \frac{\partial^2 f(p)}{\partial \tau^2} \Big|_{(z, \tau)=(0,0)} \tau \tilde{\tau} + O(\epsilon^2(|z|^4 + \tau^2)^{\frac{3}{4}}) \end{aligned}$$

where $q = (\tilde{a}_1, \dots, \tilde{a}_n, \tilde{b}_1, \dots, \tilde{b}_n, \tilde{\tau}) \in T_p(S^{2n+1})$. Observe that

$$\begin{aligned} &\frac{\partial E_f(u)}{\partial p} \cdot q + \frac{n}{n+1} E(v) A^{-\frac{2n+1}{n+1}} \text{Vol}(S^{2n+1}, \theta_0) df(p) \cdot q \\ &= -\frac{n}{n+1} E(v) A^{-\frac{2n+1}{n+1}} \int_{S^{2n+1}} (d(f \circ \phi_{-p, \epsilon}) \cdot q - df(p) \cdot q) v^{2+\frac{2}{n}} dV_{\theta_0} \\ &= -\frac{n}{n+1} E(v) A^{-\frac{2n+1}{n+1}} \left[\int_{S^{2n+1}} (d(f \circ \phi_{-p, \epsilon}) \cdot q - df(p) \cdot q) dV_{\theta_0} \right. \\ &\quad \left. + \int_{S^{2n+1}} (d(f \circ \phi_{-p, \epsilon}) \cdot q - df(p) \cdot q) (v^{2+\frac{2}{n}} - 1) dV_{\theta_0} \right] \\ &= -\frac{n}{n+1} E(v) A^{-\frac{2n+1}{n+1}} (I_1 + I_2), \end{aligned}$$

then the assertion (5.38) follows by

$$|I_1| \leq C\epsilon^2 |q| \quad \text{and} \quad |I_2| \leq C\epsilon \|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)} |q|,$$

as similarly obtained in (a).

(d) By a direct computation, we have

$$\begin{aligned}
\left\langle \frac{\partial}{\partial v} E_f(u), v-1 \right\rangle &= 2A^{-\frac{n}{n+1}} \left[\int_{S^{2n+1}} \left(\left(2 + \frac{2}{n}\right) |\nabla_{\theta_0} v|_{\theta_0}^2 + R_{\theta_0} v(v-1) \right) dV_{\theta_0} \right. \\
&\quad \left. - E(v) A^{-1} \int_{S^{2n+1}} f \circ \phi_{-p, \epsilon} v^{\frac{n+2}{n}} (v-1) dV_{\theta_0} \right] \\
&= 2A^{-\frac{n}{n+1}} \left[\int_{S^{2n+1}} R_{\theta_0} (v-1)^2 dV_{\theta_0} - \int_{S^{2n+1}} R_{\theta_0} (v^{\frac{n+2}{n}} - 1)(v-1) dV_{\theta_0} \right. \\
&\quad \left. + \left(2 + \frac{2}{n}\right) \int_{S^{2n+1}} |\nabla_{\theta_0} v|_{\theta_0}^2 dV_{\theta_0} + I \right],
\end{aligned}$$

where

$$\begin{aligned}
(5.44) \quad I &= - \int_{S^{2n+1}} (E(v) A^{-1} f \circ \phi_{-p, \epsilon} - R_{\theta_0}) v^{\frac{n+2}{n}} (v-1) dV_{\theta_0} \\
&= -(E(v) - R_{\theta_0} \text{Vol}(S^{2n+1}, \theta_0)) A^{-1} \int_{S^{2n+1}} f \circ \phi_{-p, \epsilon} v^{\frac{n+2}{n}} (v-1) dV_{\theta_0} \\
&\quad - R_{\theta_0} A^{-1} \int_{S^{2n+1}} (f \circ \phi_{-p, \epsilon} \cdot \text{Vol}(S^{2n+1}, \theta_0) - A) v^{\frac{n+2}{n}} (v-1) dV_{\theta_0} \\
&= I_1 + I_2.
\end{aligned}$$

In the following, we use the notation in the proof of Lemma 4.7. By the identity

$$(5.45) \quad v^{\frac{n+2}{n}} - 1 = \frac{n+2}{n} (v-1) + o(|v-1|),$$

and (4.15), we obtain

$$\begin{aligned}
&\int_{S^{2n+1}} R_{\theta_0} (v-1)^2 dV_{\theta_0} - \int_{S^{2n+1}} R_{\theta_0} (v^{\frac{n+2}{n}} - 1)(v-1) dV_{\theta_0} + \left(2 + \frac{2}{n}\right) \int_{S^{2n+1}} |\nabla_{\theta_0} v|_{\theta_0}^2 dV_{\theta_0} \\
&= \left(2 + \frac{2}{n}\right) \int_{S^{2n+1}} |\nabla_{\theta_0} v|_{\theta_0}^2 dV_{\theta_0} - \frac{2R_{\theta_0}}{n} \int_{S^{2n+1}} (v-1)^2 dV_{\theta_0} + o(1) \|v-1\|_{S_1^2(S^{2n+1}, \theta_0)}^2 \\
&= \left(2 + \frac{2}{n}\right) \left(\sum_{i=1}^{\infty} \lambda_i |v^i|^2 - \frac{n}{2} \sum_{i=0}^{\infty} |v^i|^2 \right) + o(1) \|v-1\|_{S_1^2(S^{2n+1}, \theta_0)}^2 \\
&\geq \left(2 + \frac{2}{n}\right) \left(\frac{\lambda_{2n+3} - n/2}{\lambda_{2n+3} + 1} \right) \sum_{i=2n+3}^{\infty} (\lambda_i + 1) |v^i|^2 + o(1) \|v-1\|_{S_1^2(S^{2n+1}, \theta_0)}^2 \\
&\geq C_0 \|v-1\|_{S_1^2(S^{2n+1}, \theta_0)}^2.
\end{aligned}$$

On the other hand, we can estimate (5.44) as follows. By (5.35), (5.45), and Lemma 5.3, we have

$$|I_1| \leq C \|v-1\|_{S_1^2(S^{2n+1}, \theta_0)}^3 = o(1) \|v-1\|_{S_1^2(S^{2n+1}, \theta_0)}^2.$$

By (4.34), (5.35), and the fact that

$$|df(p)| \rightarrow 0 \quad \text{as } r_0 \rightarrow 0,$$

we also have

$$\begin{aligned}
|I_2| &\leq C \left| \text{Vol}(S^{2n+1}, \theta_0) \int_{S^{2n+1}} (f \circ \phi_{-p, \epsilon} - f(p)) v^{\frac{n+2}{n}} (v-1) dV_{\theta_0} \right| \\
&\quad + C \left| (f(p) \text{Vol}(S^{2n+1}, \theta_0) - A) \int_{S^{2n+1}} v^{\frac{n+2}{n}} (v-1) dV_{\theta_0} \right| \\
&\leq C (\|f \circ \phi_{-p, \epsilon} - f(p)\|_{L^2(S^{2n+1}, \theta_0)} + |f(p) \text{Vol}(S^{2n+1}, \theta_0) - A|) \cdot \|v-1\|_{S_1^2(S^{2n+1}, \theta_0)} \\
&\leq (o(1)\epsilon + C\epsilon^2) \|v-1\|_{S_1^2(S^{2n+1}, \theta_0)} = o(1)\epsilon \|v-1\|_{S_1^2(S^{2n+1}, \theta_0)}.
\end{aligned}$$

Therefore, the above estimates yields (5.39). \square

Now we are going to complete the proof of Proposition 5.1 by proving part (iii) and (iv). Recall our convention in the proof of part (ii). Choose $\nu \leq r_0^3 \leq \nu_0$ and $r_0 > 0$ sufficiently small such that $B_{r_0}(p_i) \subset L_{\beta_i + \nu} \setminus L_{\beta_i - \nu}$. Similar to (ii), for any $1 \leq i \leq N$ and a sufficient large $T > 0$, we can show that $u(T, L_{\beta_i + \nu_0}) \subset L_{\beta_i + \nu}$. In addition, for any $u_0 \in L_{\beta_i + \nu_0}$, if necessary, choosing a larger $T = T(u_0) > 0$, we either have $u(T, u_0) \in L_{\beta_i - \nu_0}$ or $u(t, u_0) \in B_{r_0/4}(p_i)$ for some $t \in [0, T]$.

For $u = (\epsilon, p, \nu) \in B_{r_0}(p_i)$, we have

$$\begin{aligned}
(5.46) \quad &E_f(u) - \beta_i \\
&= \frac{E(u)}{(\int_{S^{2n+1}} f dV_{\theta})^{\frac{n}{n+1}}} - R_{\theta_0} \text{Vol}(S^{2n+1}, \theta_0)^{\frac{1}{n+1}} f(p_i)^{-\frac{n}{n+1}} \\
&= \frac{E(v)}{(\int_{S^{2n+1}} f \circ \phi_{-p, \epsilon} dV_h)^{\frac{n}{n+1}}} - R_{\theta_0} \text{Vol}(S^{2n+1}, \theta_0)^{\frac{1}{n+1}} f(p_i)^{-\frac{n}{n+1}} \\
&= A^{-\frac{n}{n+1}} \left[\left(E(v) - R_{\theta_0} \text{Vol}(S^{2n+1}, \theta_0) \right) \right. \\
&\quad \left. - R_{\theta_0} f(p_i)^{-\frac{n}{n+1}} \text{Vol}(S^{2n+1}, \theta_0)^{\frac{1}{n+1}} \left(A^{\frac{n}{n+1}} - f(p_i)^{\frac{n}{n+1}} \text{Vol}(S^{2n+1}, \theta_0)^{\frac{n}{n+1}} \right) \right]
\end{aligned}$$

where $A = \int_{S^{2n+1}} f \circ \phi_{-p, \epsilon} dV_h$. Note that

$$\begin{aligned}
&A^{\frac{n}{n+1}} - f(p_i)^{\frac{n}{n+1}} \text{Vol}(S^{2n+1}, \theta_0)^{\frac{n}{n+1}} \\
&= f(p_i)^{\frac{n}{n+1}} \text{Vol}(S^{2n+1}, \theta_0)^{\frac{n}{n+1}} \left[\left(1 + \frac{A - f(p_i) \text{Vol}(S^{2n+1}, \theta_0)}{f(p_i) \text{Vol}(S^{2n+1}, \theta_0)} \right)^{\frac{n}{n+1}} - 1 \right] \\
&= f(p_i)^{\frac{n}{n+1}} \text{Vol}(S^{2n+1}, \theta_0)^{\frac{n}{n+1}} \left[\frac{n}{n+1} \frac{A - f(p_i) \text{Vol}(S^{2n+1}, \theta_0)}{f(p_i) \text{Vol}(S^{2n+1}, \theta_0)} + O(|A - f(p_i) \text{Vol}(S^{2n+1}, \theta_0)|^2) \right]
\end{aligned}$$

and

$$A - f(p_i) \text{Vol}(S^{2n+1}, \theta_0) = A - f(p) \text{Vol}(S^{2n+1}, \theta_0) + \text{Vol}(S^{2n+1}, \theta_0) (f(p) - f(p_i)),$$

together with Lemma 5.4(a), we find that

$$\begin{aligned}
(5.47) \quad &f(p_i)^{\frac{1}{n+1}} \left[A^{\frac{n}{n+1}} - f(p_i)^{\frac{n}{n+1}} \text{Vol}(S^{2n+1}, \theta_0)^{\frac{n}{n+1}} \right] \\
&= \text{Vol}(S^{2n+1}, \theta_0)^{-\frac{1}{n+1}} \frac{n}{n+1} A_6 \epsilon^2 \Delta_{\theta_0} f(p) + \text{Vol}(S^{2n+1}, \theta_0)^{\frac{n}{n+1}} \frac{n}{n+1} (|p^+|^2 - |p^-|^2) \\
&\quad + o(1)(\epsilon^2 + |p - p_i|^2 + \|v-1\|_{S_1^2(S^{2n+1}, \theta_0)}^2).
\end{aligned}$$

Hence, from (5.46) and Lemma 5.3, we conclude that

$$E_f(u) - \beta_i \geq C\|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)} - C(\epsilon^2 + |p - p_i|^2).$$

Consequently, for $u \in L_{\beta_i+\nu} \cap B_{r_0}(p_i)$, we have

$$(5.48) \quad \|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)} \leq C(\epsilon^2 + |p - p_i|^2 + r_0^3).$$

Now we still use the same normalization (5.33) in t used in the proof of part (ii). Now with this scale, (5.47), Proposition 2.1, Lemma 4.6 and 4.8 yield that

$$\begin{aligned} \frac{d}{d\tau} E_f(U(\tau, u_0)) &= \epsilon^{-2} \frac{d}{dt} E_f(u(t(\tau), u_0)) \\ &\leq -C_3(|f'(p)|^2 + \epsilon^2 |\Delta_{\theta_0} f(p)|^2) \\ &\leq -C_4(\epsilon^2 + |p - p_i|^2), \end{aligned}$$

with uniform constants $C_3 > 0$, $C_4 > 0$ and for all $u_0 \in B_{r_0}(p_i)$. Note that the last inequality holds because with the coordinates we chose, $|f'(p)|^2 = |p - p_i|^2$ and observe that the non-degeneracy condition implies that $|\Delta_{\theta_0} f(p)| > 0$ if r_0 is sufficiently small since p_i is a critical point of f .

Thus for each $u_0 \in B_{r_0} \setminus B_{r_0/4}(p_i)$, we have

$$(5.49) \quad \frac{d}{d\tau} E_f(U(\tau, u_0)) \leq -C_5 r_0^2,$$

with a uniform constant $C_5 > 0$ in view of (5.48). Hence, transversal time of the annular region $L_{\beta_i+\nu} \cap (B_{r_0/2} \setminus B_{r_0/4}(p_i))$ is uniformly positive. Choosing sufficiently large $T^* > 0$ and sufficiently small $\nu > 0$, we have

$$(5.50) \quad U(T^*, L_{\beta_i+\nu}) \subset L_{\beta_i-\nu} \cup (B_{r_0/2}(p_i) \cap L_{\beta_i+\nu}).$$

Then

$$T_\nu(u_0) = \min\{T^*, \inf\{t : E_f(U(t, u_0)) \leq \beta_i - \nu\}\}$$

continuously depends on u_0 . Thus the map $(t, u_0) \mapsto U(\min\{t, T_\nu(u_0)\}, u_0)$ gives a homotopy equivalence of $L_{\beta_i+\nu}$ with a subset of $L_{\beta_i-\nu} \cup (B_{r_0/2}(p_i) \cap L_{\beta_i+\nu})$.

With all these preparations, now we are ready to prove part (iii) and (iv).

Proof of Proposition 5.1 (iii). Assume $\Delta_{\theta_0} f(p_i) > 0$. For $u = (\epsilon, p, v) \in B_{r_0}(p_i)$, denote the vector field $X(u)$ on $B_{r_0}(p_i)$ by setting

$$X(u) = (1, 0, 0).$$

Then let $G(u, s)$ be the solution of the flow equation

$$\frac{d}{ds} G(u, s) = X(G(u, s)),$$

with initial data $G(u, 0) = u$. Since X is transversal to $\partial B_{r_0}(p_i)$ and $G(u, r_0) \notin B_{r_0}(p_i)$, there exists a first time $0 \leq s = s(u) \leq r_0$ such that $G(u, s(u)) \notin B_{r_0}(p_i)$ and furthermore the map $u \mapsto s(u)$ is continuous. Then $H(u, s) = G(u, \min\{s, s(u)\})$ defines a homotopy $H : \overline{B_{r_0}(p_i)} \times [0, r_0] \rightarrow \overline{B_{r_0}(p_i)}$ such that

$$H(B_{r_0}(p_i), r_0) \subset \partial B_{r_0}(p_i) \quad \text{and} \quad H(\cdot, s)|_{\partial B_{r_0}(p_i)} = id, \quad 0 \leq s \leq r_0.$$

Then by (5.37), letting $u_s = H(u, s)$, we have

$$\begin{aligned} \frac{d}{ds} E_f(u_s) &= dE_f(u_s) \cdot X(u_s) = \frac{\partial}{\partial \epsilon} E_f(u_s) \\ &\leq -\frac{n^2}{2} f(p)^{-\frac{2n+1}{n+1}} \text{Vol}(S^{2n+1}, \theta_0)^{-\frac{2n}{n+1}} \epsilon A_6 \Delta_{\theta_0} f(p) + o(r_0). \end{aligned}$$

It follows that there exists a uniform constant $C_5 > 0$ such that

$$E_f(H(u, r)) \leq E_f(u) - C_5 r_0^2 \leq \beta_i - \nu \quad \text{for all } u \in B_{r_0/2}(p_i) \cap L_{\beta_i+\nu}$$

if $r_0 > 0$ is sufficiently small. Composing H with the flow $(t, u_0) \mapsto U(\min\{t, T_\nu(u_0)\}, u_0)$, we then obtain a homotopy $K : L_{\beta_i+\nu_0} \times [0, 1] \rightarrow L_{\beta_i+\nu_0}$ such that $K(L_{\beta_i+\nu_0}, 1) \subset L_{\beta_i+\nu_0}$. Moreover, by the choice of $r_0 > 0$, it is easy to show that

$$K(\cdot, r)|_{L_{\beta_i-\nu}} = id \quad \text{for } 0 \leq r \leq 1.$$

Finally, for each $u_0 \in L_{\beta_i-\nu}$, let $T_{\nu_0}(u_0) = \inf\{t \geq 0 : E_f(U(t, u_0)) \leq \beta_i - \nu_0\}$. As in the proof of part (ii), the number $T_{\nu_0}(u_0)$ are uniformly bounded and continuously depend on u_0 . By composing K with the flow $(t, u) \rightarrow U(\min\{t, T_{\nu_0}(u_0)\}, u_0)$, we therefore obtain a homotopy equivalence of $L_{\beta_i+\nu}$ with $L_{\beta_i-\nu}$. This finishes the proof of part (iii). \square

Proof of Proposition 5.1 (iv). Suppose $\Delta_{\theta_0} f(p_i) < 0$. From (5.46), (5.47), and with the constant C_2 in Lemma 5.3, we find that

$$\begin{aligned} E_f(u) - \beta_i &\geq A^{-\frac{n}{n+1}} \left[C_2 \|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)}^2 - \frac{n}{n+1} R_{\theta_0} A_6 \epsilon^2 f(p_i)^{-1} \Delta_{\theta_0} f(p) \right. \\ &\quad + \frac{n}{n+1} R_{\theta_0} \text{Vol}(S^{2n+1}, \theta_0) f(p_i)^{-1} (|p^-|^2 - |p^+|^2) \\ &\quad \left. + o(1)(\epsilon^2 + |p - p_i|^2 + \|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)}^2) \right], \end{aligned}$$

where $o(1) \rightarrow 0$ as $r_0 \rightarrow 0$. Then we deduce that there exists some number $\delta > 0$ with $4\delta^2 < \frac{7}{16} \min\{1, r_0^2\}$ such that

$$(5.51) \quad \epsilon^2 + |p^-|^2 + \|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)}^2 \leq r_0^2/4,$$

for any $u = (\epsilon, p, v) \in B_{r_0}(p_i) \cap L_{\beta_i+\nu}$ with $|p^+| < 2\delta r_0$, provided $r_0 > 0$ is sufficiently small and $\nu \leq r_0^3$.

Let $a_+ = \max\{a, 0\}$ for $a \in \mathbb{R}$. We construct a cut-off function η defined by $\eta = \eta(|p^+|) = \left(1 - \frac{(|p^+| - \delta r_0)_+}{\delta r_0}\right)_+$ with $\delta > 0$ given as above. For $0 \leq r \leq 1$, $u = (\epsilon, p, v) \in B_{r_0}(p_i)$, choose $\epsilon_0 > 0$ sufficiently small such that $0 < \frac{1}{3}\epsilon < \epsilon_0 < \frac{2}{3}\epsilon$, and define u_r by

$$u_r = (\epsilon_r, p_r, v_r) = (\epsilon + (\epsilon_0 - \epsilon)r\eta, p - r\eta p^-, ((1 - r\eta)v^{2+\frac{2}{n}} + r\eta)^{\frac{n}{2n+2}}).$$

First we claim that if $\|v - 1\|_{C_P^1(S^{2n+1})}$ is sufficiently small, then $u_r \in B_{r_0}(p_i)$. To see this, we first consider the function $g(r)$ with $\eta = 1$:

$$g(r) = (\epsilon + (\epsilon_0 - \epsilon)r)^2 + |p - rp^-|^2 + \|v_r - 1\|_{S_1^2(S^{2n+1}, \theta_0)}^2.$$

Then we have

$$\begin{aligned} g'(r) &= 2(\epsilon + (\epsilon_0 - \epsilon)r)(\epsilon_0 - \epsilon) + 2\langle p - rp^-, -p^- \rangle \\ &\quad + 2 \int_{S^{2n+1}} \left[\langle \nabla_{\theta_0} v_r, \nabla_{\theta_0} \frac{dv_r}{dr} \rangle_{\theta_0} + (v_r - 1) \frac{dv_r}{dr} \right] dV_{\theta_0}. \end{aligned}$$

A simple calculation gives $\frac{d^2 v_r}{dr^2} = -\frac{n+2}{n}v_r^{-1} \left(\frac{dv_r}{dr}\right)^2$. Thus we have

$$\begin{aligned}
g''(r) &= 2(\epsilon_0 - \epsilon)^2 + 2|p^-|^2 + 2 \int_{S^{2n+1}} \left[\left| \nabla_{\theta_0} \frac{dv_r}{dr} \right|_{\theta_0} + \left(\frac{dv_r}{dr} \right)^2 \right] dV_{\theta_0} \\
&\quad + 2 \int_{S^{2n+1}} \left[\langle \nabla_{\theta_0} v_r, \nabla_{\theta_0} \frac{d^2 v_r}{dr^2} \rangle_{\theta_0} + (v_r - 1) \frac{d^2 v_r}{dr^2} \right] dV_{\theta_0} \\
&= 2(\epsilon_0 - \epsilon)^2 + 2|p^-|^2 + 2 \int_{S^{2n+1}} \left[\left| \nabla_{\theta_0} \frac{dv_r}{dr} \right|_{\theta_0} + \left(\frac{dv_r}{dr} \right)^2 \right] dV_{\theta_0} \\
&\quad - \frac{2(n+2)}{n} \int_{S^{2n+1}} \left[\langle \nabla_{\theta_0} v_r, \nabla_{\theta_0} \left(v_r^{-1} \left(\frac{dv_r}{dr} \right)^2 \right) \rangle_{\theta_0} + (v_r - 1) v_r^{-1} \left(\frac{dv_r}{dr} \right)^2 \right] dV_{\theta_0} \\
&= 2(\epsilon_0 - \epsilon)^2 + 2|p^-|^2 + 2 \int_{S^{2n+1}} \left[\left| \nabla_{\theta_0} \frac{dv_r}{dr} \right|_{\theta_0} + \left(\frac{dv_r}{dr} \right)^2 \right] dV_{\theta_0} \\
&\quad + \frac{2(n+2)}{n} \int_{S^{2n+1}} |\nabla_{\theta_0} v_r|_{\theta_0}^2 v_r^{-2} \left(\frac{dv_r}{dr} \right)^2 dV_{\theta_0} \\
&\quad - \frac{2(n+2)}{n} \int_{S^{2n+1}} \left[\frac{n}{n+1} v_r^{-(2+\frac{2}{n})} \frac{dv_r}{dr} \langle \nabla_{\theta_0} v_r^{2+\frac{2}{n}}, \nabla_{\theta_0} \frac{dv_r}{dr} \rangle_{\theta_0} + (v_r - 1) v_r^{-1} \left(\frac{dv_r}{dr} \right)^2 \right] dV_{\theta_0}.
\end{aligned}$$

Now observe that $|v_r^{-(2+\frac{2}{n})} \nabla_{\theta_0} v_r^{2+\frac{2}{n}}|_{\theta_0} = o(1)$ and $(v_r - 1)v_r^{-1} = o(1)$ if $\|v - 1\|_{C_P^1(S^{2n+1})}$ is sufficiently small. Hence, using the Hölder's and Young's inequality, we get

$$g''(r) \geq 2(\epsilon_0 - \epsilon)^2 + 2|p^-|^2 + (2 + o(1)) \int_{S^{2n+1}} \left[\left| \nabla_{\theta_0} \frac{dv_r}{dr} \right|_{\theta_0} + \left(\frac{dv_r}{dr} \right)^2 \right] dV_{\theta_0} \geq 0.$$

This shows that $g''(r) \geq 0$ for all $r \in [0, 1]$. Thus we conclude that

$$g(r) \leq \max\{g(0), g(1)\} = \max\{r_0^2, \epsilon_0^2 + |p^+|^2\}.$$

Now note that if $\eta r > 0$, then $\eta > 0$, hence $|p^+| < 2\delta r_0$ by the definition of η . Thus by the estimate (5.51), we have $\epsilon^2 \leq \frac{r_0^2}{4}$. This implies that $\epsilon_0^2 \leq \frac{r_0^2}{9}$ and $4\delta^2 \leq 1 - \frac{2}{9}$, we have $g(1) \leq \epsilon_0^2 + |p^+|^2 \leq (\frac{1}{9} + 4\delta^2)r_0^2 \leq r_0^2$. Therefore, we have $g(\eta r) \leq \max\{g(0), g(1)\} \leq r_0^2$.

Therefore under smallness condition of $\|u - 1\|_{C_P^1(S^{2n+1})}$, (which can be guaranteed by the construction of homotopies below), one has shown that the homotopy $H_1 : \overline{B_{r_0}(p_i)} \cap L_{\beta_i+\nu} \times [0, 1] \rightarrow \overline{B_{r_0}(p_i)}$ given by $H_1(u, r) = u_r$ is well defined and $H_1(\cdot, 1)$ maps the set $\{u \in B_{r_0}(p_i) \cap L_{\beta_i+\nu} : |p^+| < \delta r_0\}$ to the set $B_{\delta r_0}^+$, where for $0 < \rho < r_0$,

$$B_\rho^+ := \{u \in B_{r_0}(p_i) : \epsilon = \epsilon_0, p^- = 0, |p^+| < \rho, v = 1\}$$

which is diffeomorphic to the unit ball of dimension $2n + 1 - \text{ind}(f, p_i)$.

Now we need to show that the energy level of u_r is under control; that is, $E_f(u_r) \leq \beta_i + \nu$ if ν is sufficiently small. To do this, we observe that

$$\begin{aligned}
 \frac{d}{dr} E_f(u_r) &= \eta \left(\frac{\partial E_f(u_r)}{\partial \epsilon_r} (\epsilon_0 - \epsilon) - \frac{\partial E_f(u_r)}{\partial p_r} p^- \right. \\
 &\quad \left. - \frac{n}{2n+2} \left\langle \frac{\partial E_f(u_r)}{\partial v_r}, v_r^{-\frac{n+2}{n}} (v^{2+\frac{2}{n}} - 1) \right\rangle \right) \\
 (5.52) \quad &= \eta(1-r\eta)^{-1} \left(\frac{\partial E_f(u_r)}{\partial \epsilon_r} (\epsilon_0 - \epsilon_r) - \frac{\partial E_f(u_r)}{\partial p_r} p^- \right. \\
 &\quad \left. - \frac{n}{2n+2} \left\langle \frac{\partial E_f(u_r)}{\partial v_r}, v_r^{-\frac{n+2}{n}} (v^{2+\frac{2}{n}} - 1) \right\rangle \right) \\
 &:= \eta(1-r\eta)^{-1} D := \eta(1-r\eta)^{-1} (I - II - III).
 \end{aligned}$$

First we deal with the last term:

$$\begin{aligned}
 III &:= \frac{n}{2n+2} \left\langle \frac{\partial E_f(u_r)}{\partial v_r}, v_r^{-\frac{n+2}{n}} (v^{2+\frac{2}{n}} - 1) \right\rangle \\
 &= \frac{n}{n+1} A_r^{-\frac{n}{n+1}} \left[\int_{S^{2n+1}} (2 + \frac{2}{n}) \langle \nabla_{\theta_0} v_r, \nabla_{\theta_0} (v_r^{-\frac{n+2}{n}} (v^{2+\frac{2}{n}} - 1)) \rangle_{\theta_0} \right. \\
 &\quad \left. + \int_{S^{2n+1}} R_{\theta_0} v_r v_r^{-\frac{n+2}{n}} (v^{2+\frac{2}{n}} - 1) dV_{\theta_0} \right] \\
 &\quad - \frac{n}{n+1} A_r^{-\frac{n}{n+1}} E(v_r) A_r^{-1} \int_{S^{2n+1}} f \circ \phi_{-p_r, \epsilon_r} (v_r^{2+\frac{2}{n}} - 1) dV_{\theta_0} \\
 &:= \frac{n}{n+1} A_r^{-\frac{n}{n+1}} (III_1 + III_2),
 \end{aligned}$$

where $A_r = \int_{S^{2n+1}} f \circ \phi_{-p_r, \epsilon_r} v_r^{2+\frac{2}{n}} dV_{\theta_0}$. Since $\|v-1\|_{C_P^1(S^{2n+1})} = o(1)$ is sufficiently small, it yields $\|v_r - 1\|_{C_P^1(S^{2n+1})} = o(1)$. We can estimate III_1 as follows:

$$\begin{aligned}
 III_1 &= (2 + \frac{2}{n})^2 \int_{S^{2n+1}} |\nabla_{\theta_0} v_r|_{\theta_0}^2 dV_{\theta_0} + R_{\theta_0} \int_{S^{2n+1}} (v_r^{\frac{2}{n}} - 1) (v_r^{2+\frac{2}{n}} - 1) dV_{\theta_0} \\
 &\quad - \frac{(2n+2)(n+2)}{n^2} \int_{S^{2n+1}} |\nabla_{\theta_0} v_r|_{\theta_0}^2 (1 - v_r^{-(2+\frac{2}{n})}) dV_{\theta_0} \\
 &= (2 + \frac{2}{n})^2 \left[\int_{S^{2n+1}} |\nabla_{\theta_0} v_r|_{\theta_0}^2 dV_{\theta_0} - \frac{n}{2} \int_{S^{2n+1}} (v_r - 1)^2 dV_{\theta_0} \right] \\
 &\quad + o(1) \|v_r - 1\|_{S_1^2(S^{2n+1}, \theta_0)}^2.
 \end{aligned}$$

where in the first equality we have used the fact that $\int_{S^{2n+1}} (v_r^{2+\frac{2}{n}} - 1) dV_{\theta_0} =$

$(1-r\eta) \int_{S^{2n+1}} (v^{2+\frac{2}{n}} - 1) dV_{\theta_0} = 0$. Observe that the following estimate holds true

by the same argument of proving (4.10):

$$(5.53) \quad \int_{S^{2n+1}} |\nabla_{\theta_0} v_r|_{\theta_0}^2 dV_{\theta_0} - \frac{n}{2} \int_{S^{2n+1}} (v_r - 1)^2 dV_{\theta_0} \geq \left[\lambda_{2n+3} - \frac{n}{2} + o(1) \right] \|v_r - 1\|_{S_1^2(S^{2n+1}, \theta_0)}^2,$$

which will be used in controlling D . For III_2 , we can rewrite it as

$$III_2 = -E(v_r) \left(1 - A_r^{-1} \int_{S^{2n+1}} f \circ \phi_{-p_r, \epsilon_r} dV_{\theta_0} \right).$$

Note that

$$\begin{aligned}
& \int_{S^{2n+1}} f \circ \phi_{-p_r, \epsilon_r} dV_{\theta_0} - f(p_r) \text{Vol}(S^{2n+1}, \theta_0) \\
&= (A_r - f(p_r) \text{Vol}(S^{2n+1}, \theta_0)) + \int_{S^{2n+1}} f \circ \phi_{-p_r, \epsilon_r} (1 - v_r^{2+\frac{2}{n}}) dV_{\theta_0} \\
&= (A_r - f(p_r) \text{Vol}(S^{2n+1}, \theta_0)) + \int_{S^{2n+1}} (f \circ \phi_{-p_r, \epsilon_r} - f(p_r)) (1 - v_r^{2+\frac{2}{n}}) dV_{\theta_0} \\
&= (A_r - f(p_r) \text{Vol}(S^{2n+1}, \theta_0)) + o(1) \epsilon_r \|v_r - 1\|_{S_1^2(S^{2n+1}, \theta_0)}
\end{aligned}$$

where the third equality follows from $\int_{S^{2n+1}} (v_r^{2+\frac{2}{n}} - 1) dV_{\theta_0} = 0$ and the last equality follows from (5.40). Hence,

$$\begin{aligned}
& 1 - A_r^{-1} \int_{S^{2n+1}} f \circ \phi_{-p_r, \epsilon_r} dV_{\theta_0} \\
&= A_r^{-1} \left[A_r - f(p_r) \text{Vol}(S^{2n+1}, \theta_0) - \left(\int_{S^{2n+1}} f \circ \phi_{-p_r, \epsilon_r} dV_{\theta_0} - f(p_r) \text{Vol}(S^{2n+1}, \theta_0) \right) \right] \\
&= o(1) \epsilon_r \|v_r - 1\|_{S_1^2(S^{2n+1}, \theta_0)}.
\end{aligned}$$

Combining these estimates, we obtain

$$\begin{aligned}
III &= 2(2 + \frac{2}{n}) A_r^{-\frac{n}{n+1}} \left[\int_{S^{2n+1}} |\nabla_{\theta_0} v_r|_{\theta_0}^2 dV_{\theta_0} - \frac{n}{2} \int_{S^{2n+1}} (v_r - 1)^2 dV_{\theta_0} \right] \\
&\quad + o(1) (\epsilon_r + \|v_r - 1\|_{S_1^2(S^{2n+1}, \theta_0)}) \|v_r - 1\|_{S_1^2(S^{2n+1}, \theta_0)}.
\end{aligned}$$

For I and II , we can apply Lemma 5.4 to get

$$II = -\frac{n}{n+1} A_r^{-\frac{n}{n+1}} E(v_r) \text{Vol}(S^{2n+1}, \theta_0)^{\frac{2n+1}{n+1}} \frac{df(p_r) \cdot p_r^-}{A_r} + C \epsilon_r (\epsilon_r + \|v_r - 1\|_{S_1^2(S^{2n+1}, \theta_0)}) |p_r^-|$$

and

$$\begin{aligned}
I &= -\frac{n}{n+1} A_r^{-\frac{n}{n+1}} E(v_r) \epsilon_r A_6 \frac{\Delta_{\theta_0} f(p_r)}{A_r} (\epsilon_0 - \epsilon_r) \\
&\quad + C (\epsilon_r^2 + (\epsilon_r + |p_r - p_i|) \|v_r - 1\|_{S_1^2(S^{2n+1}, \theta_0)}) (\epsilon_r - \epsilon_0).
\end{aligned}$$

Note that in the local coordinates of p_i , $f(p_r) = f(p_i) + |p_r^+|^2 - |p_r^-|^2$ and $df(p_r) \cdot p_r^- = -2|p_r^-|^2$. Therefore, combining the estimates of I , II , III and (5.53), we obtain

$$\begin{aligned}
D &\leq A_r^{-\frac{n}{n+1}} \left\{ -\frac{n A_6}{n+1} E(v_r) \epsilon_r (\epsilon_0 - \epsilon_r) \frac{\Delta_{\theta_0} f(p_r)}{A_r} - \frac{2n}{n+1} E(v_r) \text{Vol}(S^{2n+1}, \theta_0)^{\frac{2n+1}{n+1}} \frac{|p_r^-|^2}{A_r} \right. \\
&\quad + C |p_r^+| \|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)} (\epsilon_r - \epsilon_0) - 2(2 + \frac{2}{n}) \left[\lambda_{2n+3} - \frac{n}{2} + o(1) \right] \|v_r - 1\|_{S_1^2(S^{2n+1}, \theta_0)}^2 \\
&\quad \left. + o(1) (\epsilon_r (\epsilon_r - \epsilon_0) + \|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)}^2 + |p_r^-|^2) \right\}.
\end{aligned}$$

Now set

$$d = \min \left\{ \min_{|p_r - p_i| \leq r_0} \left(-\frac{n A_6}{n+1} E(v_r) \frac{\Delta_{\theta_0} f(p_r)}{A_r} \right), \frac{2n}{n+1} \frac{E(v_r)}{A_r}, 2(2 + \frac{2}{n}) (\lambda_{2n+3} - \frac{n}{2}) \right\}.$$

Since $\Delta_{\theta_0} f(p_i) < 0$, by continuity, when r_0 is sufficiently small, $\Delta_{\theta_0} f(p_r) < 0$ if $|p_r - p_i| < r_0$. Note also that $A_6 > 0$ by (5.41), we have $d > 0$. We can rewrite the

above estimate as

$$(5.54) \quad \begin{aligned} D \leq A_r^{-\frac{n}{n+1}} & \left\{ -\frac{d}{2}(\epsilon_r(\epsilon_r - \epsilon_0) + \|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)}^2 + |p_r^-|^2) \right. \\ & \left. + C|p_r^+| \|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)}(\epsilon_r - \epsilon_0) \right\} := A_r^{-\frac{n}{n+1}} D_1. \end{aligned}$$

If $\eta > 0$, then by the definition of η we have $|p^+| < 2\delta r_0$. By definition of p_r , $|p_r^+| = |p^+|$, hence, we have $|p_r^+| < 2\delta r_0$. Now if we choose r_0 sufficiently small such that $2C\delta r_0 < d$, we have

$$D_1 \leq -\frac{d}{2} \left\{ (\epsilon_r(\epsilon_r - \epsilon_0) + \|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)}^2 + |p_r^-|^2) - 2\|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)}(\epsilon_r - \epsilon_0) \right\}.$$

Since $\epsilon_r(\epsilon_r - \epsilon_0) \geq (\epsilon_r - \epsilon_0)^2$, it is easy to see that $D_1 < 0$ when $\|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)}$ is sufficiently small, which implies that $D < 0$ by (5.54). Hence, by (5.52), we have $\frac{d}{dr} E_f(u_r) \leq 0$. Note that when $r = 0$, $u_r = u = (\epsilon, p, v)$. Since $u \in B_{r_0}(p_i) \cap L_{\beta_i + \nu}$, we have $E_f(u_r) \leq E_f(u) \leq \beta_i + \nu$ for all $r \in [0, 1]$.

Moreover, from (5.51) and our choice of δ , we have

$$H(\cdot, r)|_{\partial B_{r_0}(p_i) \cap L_{\beta_i + \nu}} = id, \quad \text{for } 0 \leq r \leq 1.$$

Denote the vector field $X_1(u)$ as

$$X_1(u) = (0, p^+, 0),$$

and $G_1(u, s)$ solves the flow equation

$$\frac{d}{ds} G_1(u, s) = X_1(G_1(u, s)), \quad 0 \leq s \leq \delta^{-1},$$

with initial data $G_1(u, 0) = u$. Notice that X_1 is transversal to $\partial B_{r_0}(p_i)$ within $L_{\beta_i + \nu}$; in addition, for any $u \in B_{r_0}(p_i) \cap L_{\beta_i + \nu}$ with $|p^+| \geq \delta r_0$, there holds $G_1(u, \delta^{-1}) \notin B_{r_0}(p_i)$ for some sufficiently small $\delta > 0$, then there exists a first time $0 \leq s_1 \leq r(u) \leq \delta^{-1}$ such that $G_1(u, s_1) \notin B_{r_0}(p_i)$, and the map $u \mapsto s_1(u)$ is continuous. We extend this map to whole set $B_{r_0}(p_i) \cap L_{\beta_i + \nu}$ by letting $s_1(u) = \delta^{-1}$ whenever $G_1(u, s) \in B_{r_0}(p_i)$ for all $s \in [0, r_1]$. Setting $H_2(u, s) = G_1(u, \min\{s, s_1(u)\}) = u_s$, with a uniform $C > 0$, we obtain by (5.38) that

$$\begin{aligned} \frac{dE_f(u_s)}{ds} &= \frac{\partial E_f(u_s)}{\partial p} \cdot p^+ \\ &\leq -\frac{n}{n+1} E(u_s) f(p)^{-\frac{2n+1}{n+1}} df(p) \cdot p^+ + C\epsilon(\epsilon + \|v - 1\|_{S_1^2(S^{2n+1}, \theta_0)})|p^+| \\ &\leq -\frac{2n}{n+1} R_{\theta_0} \text{Vol}(S^{2n+1}, \theta_0) f(p)^{-\frac{2n+1}{n+1}} |p^+|^2 + Cr_0^3 \leq -Cr_0^2 \end{aligned}$$

if $|p^+| \geq \delta r_0$. Here we have used (4.83) in the second inequality. Then, let H be the composition of H_1 with H_2 , for sufficiently small $r_0 > 0$, it yields a homotopy $H : \overline{B_{r_0}(p_i)} \cap L_{\beta_i + \nu} \times [0, 1] \rightarrow \overline{B_{r_0}(p_i)} \cap L_{\beta_i + \nu}$ such that

$$B_{r_0}(p_i) \cap L_{\beta_i + \nu} \subset B_{\delta r_0}^+ \cup (\partial B_{r_0}(p_i) \cap L_{\beta_i + \nu})$$

and

$$H(\cdot, r)|_{\partial B_{r_0}(p_i) \cap L_{\beta_i + \nu}} = id, \quad 0 \leq r \leq 1.$$

Composing H with $U(T, \cdot)$ where $T = T(u_0) = \inf\{t \geq 0 : E_f(U(t, u_0)) \leq \beta_i - \nu\}$ for $u \in L_{\beta_i + \nu}$. From (5.49) and (5.50), since the transversal time of the annular region $L_{\beta_i + \nu} \cap (B_{r_0}(p_i) \setminus B_{r_0/4}(p_i))$ is uniformly positive, then it follows that

$U(T, \partial B_{r_0}(p_i) \cap L_{\beta_i+\nu}) \subset L_{\beta_i-\nu}$. Therefore, the proof can be followed as in part (iii). \square

This completes the proof of Proposition 5.1.

6. CONCLUDING REMARKS

We have proved that Theorem 1.4 is true when $n \geq 2$. The natural question would be: is Theorem 1.4 true when $n = 1$? We conjecture the answer is yes. In fact, we only used the assumption that $n \geq 2$ in section 4.1. Especially, we need to use the assumption $n \geq 2$ in the proof of Lemma 4.3 and 4.8. So if one can prove the results in section 4.1 for the case when $n = 1$, one would be able to prove Theorem 1.4 by following the same arguments of the remaining part of this paper.

One would also like to study the largest possible number δ_n in the simple bubble condition

$$\max_{S^{2n+1}} f / \min_{S^{2n+1}} f < \delta_n$$

such that Theorem 1.4 holds. In Theorem 1.4, we have $\delta_n = 2^{\frac{1}{n}}$. Is it the best possible?

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